M1170 MOCK EXAM Solutions

1[5] $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\left[\frac{1}{x+h+1}\right]-\left[\frac{1}{x+1}\right]}{h}$

$= \dfrac{[x+1]-[x+h+1]}{h[x+1][x+h+1]}$

$= \dfrac{-\cancel{h}}{\cancel{h}[x+1][x+h+1]}$

$\boxed{= -\dfrac{1}{(x+1)(x+h+1)}}$

2a[3] R = (price) x (number of customers)

$= x\cdot(120-2(x-10)4)$ since it loses 4 customer for every .5 increase in price.

$\boxed{R = x(200-8x)}$

b[2] P = Revenue − Overhead − (cost / customer)(no of custom

$= x(200-8x) - 100 - 4(200-8x)$

$\boxed{P = -8x^2 + 232x - 900}$

c[2] The maximum occur at the vertex, x coordinate of vertex $= \dfrac{-b}{2a} = \dfrac{-232}{-16} = 14.5$

$\boxed{\therefore \text{The optimum price} = \$14.50}$

3a [4] $\sqrt{x+1} + \sqrt{x-2} = 2$

$\sqrt{x+1} = 2 - \sqrt{x-2}$ (square both sides)

$\cancel{x}+1 = 4 - 4\sqrt{x-2} + (\cancel{x}-2)$ ✓

$\frac{-1}{-4} = \sqrt{x-2}$

$\Rightarrow x - 2 = \frac{1}{16}$

$\boxed{x = \frac{33}{16}}$ ✓✓

check (this is a <u>necessary</u> step, because the solution was obtained by squaring both sides)

$\sqrt{\frac{33}{16}+1} + \sqrt{\frac{33}{16}-2} \stackrel{?}{=} 2$

$\sqrt{\frac{49}{16}} + \sqrt{\frac{1}{16}} \stackrel{?}{=} 2$

$\frac{7}{4} + \frac{1}{4} = 2$ ✓

$\therefore \boxed{x = \frac{33}{16}}$

3b [4] $1 - \sin(t) = \sqrt{3}\cos(t)$

Here, we need to convert the equation to one involving only 1 trig. function. let's convert the cos into sin, but first we have to square both sides

$$[1 - \sin t]^2 = [\sqrt{3}\cos t]^2 \checkmark$$

$$1 - 2\sin t + \sin^2 t = 3\cos^2 t$$

$$\sin^2 t - 2\sin t + 1 = 3(1 - \sin^2 t)$$

$$4\sin^2 t - 2\sin t - 2 = 0$$

$$2\sin^2 t - \sin t - 1 = 0 \checkmark$$

$$(2\sin t + 1)(\sin t - 1) = 0$$

$\rightarrow \sin t = -\frac{1}{2}$ or $\sin t = 1$

$t = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\quad t = \frac{\pi}{2} \checkmark$

Check,

at $t = \frac{7\pi}{16}$ $\quad 1 - \sin\frac{7\pi}{6} \stackrel{?}{=} \sqrt{3}\cos\frac{7\pi}{6}$

$$1 - \left(-\frac{1}{2}\right) \stackrel{?}{=} \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)$$

$$\frac{3}{2} \neq -\frac{3}{2}$$

$\therefore t = \frac{7\pi}{6}$ is an extraneous solution at $t = \frac{11\pi}{6}$ $\quad 1 - \sin\left(\frac{11\pi}{6}\right) \stackrel{?}{=} \sqrt{3}\cos\left(\frac{11\pi}{6}\right)$

$$1 - \left(-\tfrac{1}{2}\right) \stackrel{?}{=} \sqrt{3}\left(\tfrac{\sqrt{3}}{2}\right)$$

$$\tfrac{3}{2} = \tfrac{3}{2}$$

at $t = \frac{\pi}{2}$ $\quad 1 - \sin\left(\frac{\pi}{2}\right) \stackrel{?}{=} \sqrt{3}\cos\left(\frac{\pi}{2}\right)$

$$1 - 1 \stackrel{?}{=} \sqrt{3}\,(0)$$

$$0 = 0$$

$\therefore \boxed{t = \left\{\frac{11\pi}{6}, \frac{\pi}{2}\right\}}$

3c [4] $\quad e^{x} + e^{-x} = \cancel{4}\,1$ $\quad$ (mult. by e^{x})

$e^{2x} + 1 = \cancel{4}\, e^{x}$ $\quad$ let $y = e^{x}$

$y^2 + 1 = y$

$y^2 - y + 1 = 0$

$$y = \frac{1 \pm \sqrt{1-4}}{2}$$

$\Rightarrow$ there are no real solutions for y

$\therefore$ there are no real solutions for x.

3d [4] $\cos t - \sin 2t = 0$

here, we need to uniformize the arguement. Convert $\sin 2t$ into trig. function of t. use double angle formula.

$\cos t - \sin 2t = 0$

$\cos t - 2\sin t \cos t = 0$ ✓

$\cos t \, (1 - 2\sin t) = 0$ ✓

$\Rightarrow \cos t = 0$ or $\sin t = \frac{1}{2}$ ✓

$t = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad$ $\sin t = \frac{\pi}{6}, \frac{5\pi}{6}$ ✓

$\therefore \boxed{t = \left\{ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \right\}}$

Note: here we don't have to check the solutions, because we did not arrive at them by squaring both sides of the equation.

4a [6] $f(x) = \dfrac{3x^3 - 4x^2 + x}{2x^2 + 5x - 3} = \dfrac{x(3x^2 - 4x + 1)}{(2x - 1)(x + 3)}$

$$= \frac{x(x-1)(3x-1)}{(2x-1)(x+3)}$$ ✓

$\therefore$ $\boxed{\text{zeros at } x = 0, 1, \tfrac{1}{3}}$ ✓

$\boxed{\text{vertical asymptotes at } x = -3, \tfrac{1}{2}}$ ✓

The degree of the numerator (= 3) is 1 ~~more than~~ greater than the degree of the denominator (= 2).
$\therefore$ the graph of f(x) has an oblique (slant) asymptote.

$\frac{3}{2}x - \frac{23}{4} \longrightarrow$ equation of slant asymptote.

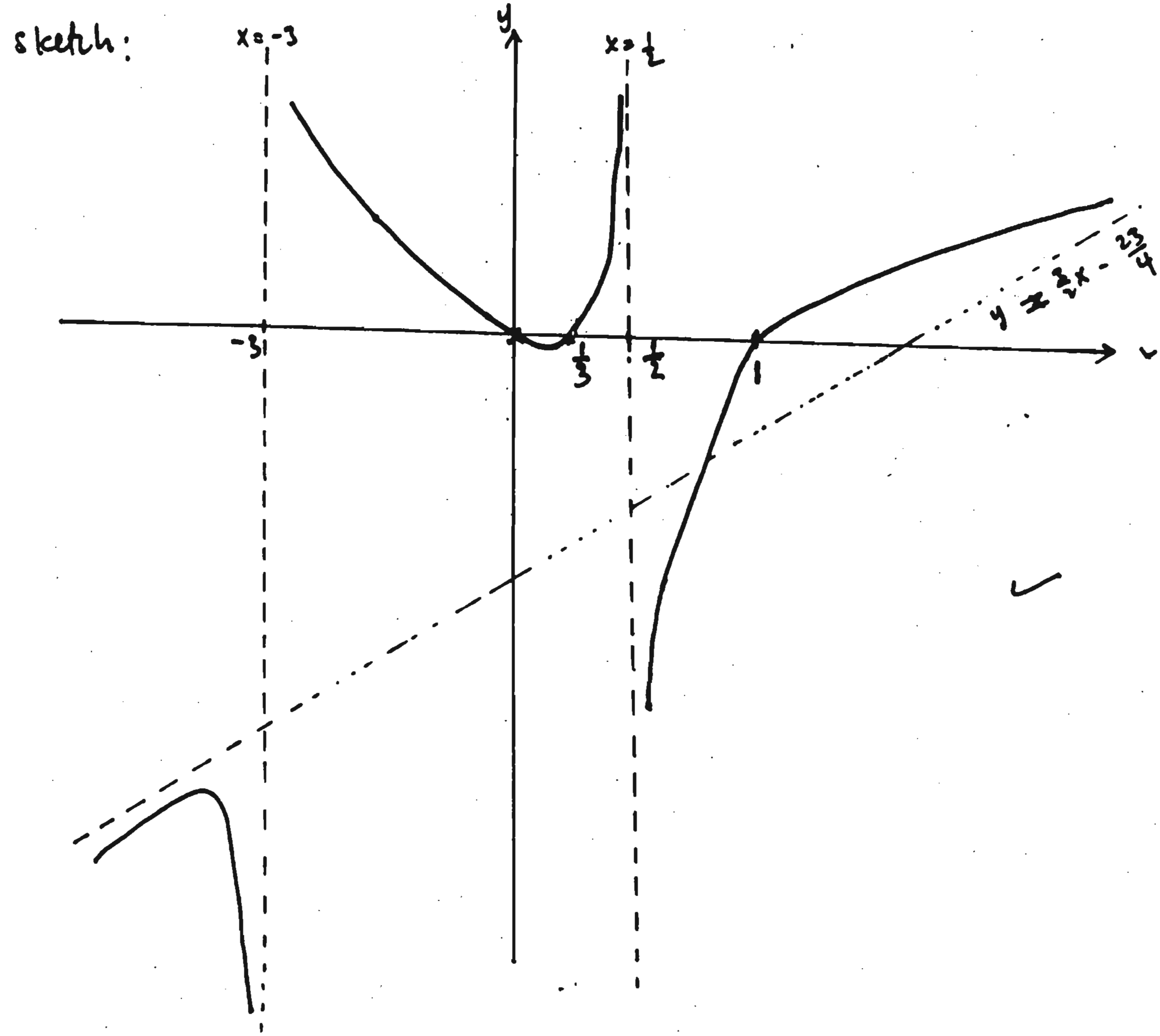
$\hookrightarrow y = \frac{3}{2}x - \frac{23}{4}$

$$
\begin{array}{r|l}
 & \frac{3}{2}x - \frac{23}{4} \\
2x^2 + 5x - 3 & 3x^3 - 4x^2 + x \\
 & 3x^3 + \frac{15}{2}x^2 - \frac{9}{2}x \quad - \\
\hline
 & -\frac{23}{2}x^2 + \frac{11}{2}x \\
 & -\frac{23}{2}x^2 - \frac{115}{4}x + \frac{69}{4} \\
\hline
 & \frac{137}{4}x - \frac{69}{4}
\end{array}
$$

polarity chart:

	$x<-3$	$-3<x<0$	$0<x<\frac{1}{3}$	$\frac{1}{3}<x<\frac{1}{2}$	$\frac{1}{2}<x<1$	$x>1$
numerator	−	−	+	−	−	+
denominator	+	−	−	−	+	+
f(x)	−	+	−	+	−	+

sketch:



b[2] $f^{-1}(x)$ does not exist, because there exist a horizontal line that crosses f(x) more than once

7

5a [c] $\cancel{y}\ x = \dfrac{1}{1+e^y}$ $\qquad$ $y\ x = \dfrac{1}{}$

$\Rightarrow x + xe^y = 1$

$e^y = \dfrac{\cancel{1} - x}{x}$

$y = \ln\left[\dfrac{1-x}{x}\right]$

$\therefore \boxed{f^{-1}(x) = \ln\left[\dfrac{1-x}{x}\right]}$

6[2] $f^{-1}(x) = \ln\left[\dfrac{1-x}{x}\right]$ is defined if $\dfrac{1-x}{x} > 0$

$\Rightarrow \cancel{x}\ 0 < x < 1$ (If you don't get this, construct a polarity chart of $\dfrac{1-x}{x}$).

$\therefore$ $\boxed{\text{domain of } f^{-1}(x) = (0, 1).}$

c[4] $\ln\left[\frac{1-x}{x}\right]$ is undefined for $x = 0, 1$

$\therefore$ there are vertical asymptotes at $x = 0, 1$.

- The range of $f^{-1}(x)$ is the domain of $f(x)$, which is all ~~ℝ~~ Reals. $\therefore$ $f^{-1}(x)$ has no horizontal asymptotes.

- x intercept occur at:

$$0 = \ln\left[\frac{1-x}{x}\right]$$

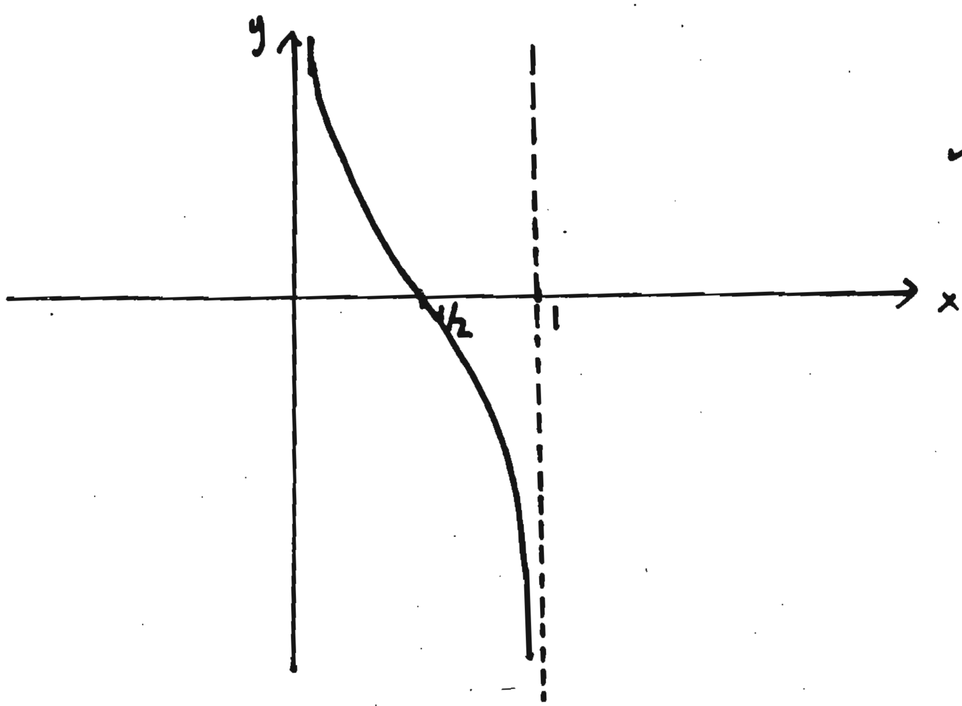
$$\Rightarrow \frac{1-x}{x} = 1$$

$$\boxed{x = \tfrac{1}{2}}$$

- as $x \to 0^+$, $\frac{1-x}{x} \to \infty$, $\ln\left[\frac{1-x}{x}\right] \to \infty$

as $x \to 1^-$, $\frac{1-x}{x} \to 0$, $\ln\left[\frac{1-x}{x}\right] \to -\infty$

$\therefore$ Sketch:



6a [4] after 1 hr, the amount left $= A_0(.75)$

after 2 hr. the amount left $= A_0(.75)^2$

$$\therefore \boxed{A(t) = A_0(.75)^t}$$

b [2] $A(t) = A_0(.75)^t = \frac{1}{2}A_0$

$$(.75)^t = \frac{1}{2}$$

$$\ln (.75)^t = \ln \frac{1}{2}$$

$$t \ln (.75) = \ln \frac{1}{2}$$

$$t = \frac{\ln .5}{\ln .75}$$

$$\boxed{t \simeq 2.41}$$

The 'half life' of the drug is 2.41 hours, or $\simeq$ 2 hours and 25 minutes.

7a [4] $T(t) = 98.6 + A\sin(bt + c)$

period $= \frac{2\pi}{b} = 24 \Rightarrow \boxed{b = \frac{\pi}{12}}$ ✓

amplitude = half the distance of max. and min.

$= \frac{98.9 - 98.6}{2} = 0.3 \Rightarrow \boxed{A = 0.3}$ ✓

at 5 am, temperature = 98.6

$\Rightarrow T(5) = 98.6 + 0.3\sin\left(\frac{\pi}{12}(5) + c\right) = 98.3$

$\Rightarrow \sin\left[\frac{5\pi}{12} + c\right] = -1$

$\Rightarrow \frac{5\pi}{12} + c = -\frac{\pi}{2}$

$\Rightarrow \boxed{c = \frac{-11\pi}{12}}$ ✓✓

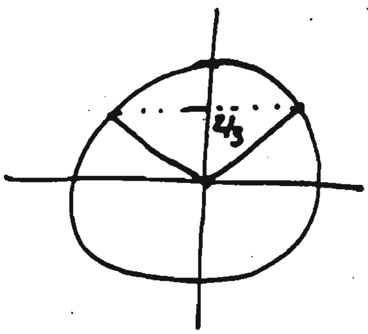
$\therefore \boxed{T(t) = 98.6 + 0.3\sin\left(\frac{\pi}{12}t - \frac{4\pi}{12}\right)}$

1b [4] We want an interval of t for which

$T(t) > 98.8$

i.e.: $98.6 + 0.3 \sin\left(\frac{\pi}{12}t - \frac{11\pi}{12}\right) > 98.8$

$\Rightarrow \quad \sin\left(\frac{\pi}{12}t - \frac{11\pi}{12}\right) > \frac{2}{3}$



$\sin^{-1}\left(\frac{2}{3}\right) = .73$

$\therefore \cancel{\frac{\pi}{12}}\ .73 < \frac{\pi}{12}t - \frac{11\pi}{12} < \pi - .73$

$\cancel{.73\ \frac{12}{\pi} < t}$

$\left(.73 + \frac{11\pi}{12}\right)\frac{12}{\pi} < t < \left(\pi - .73 + \frac{11\pi}{12}\right)\frac{12}{\pi}$

$13.8 < t < 20.2$

$\therefore$ The percentage of time that T exceed 98.8°F

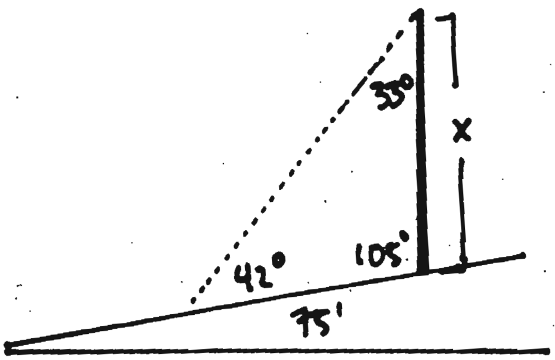
$= \frac{20.2 - 13.8}{24} \approx \underline{\underline{27\%}}.$

8 [5]. - the angle between the road and the dashed line

$= 57° - 15° = 42°$.

- the angle between the bottom of the pole and the road

$= 90° + 15° = 105°$.



. the angle between the top of the pole to the dashed line $= 180° - 42° - 105° = 33°$.

$\therefore$ (using law of sines):

$$\frac{\sin 42°}{x} = \frac{\sin 33°}{75}$$

$$\Rightarrow x = \frac{(\sin 42°)(75)}{\sin 33°} = 92.14'$$

$\therefore$ $\boxed{\text{The pole is } 92.14'}$

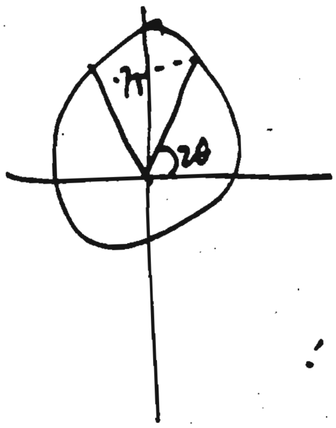
7 [4] $K = \frac{v^2}{16} \sin\theta \cos\theta = 150$

$\Rightarrow \sin\theta \cos\theta = .375$ ✓

Now we need to reduce the equation into one with only one trig. function. use double angle formula : $\sin 2\theta = 2 \sin\theta \cos\theta$. Multiply above eqn' by 2.

$\rightarrow 2 \sin\theta \cos\theta = .75$

$\Rightarrow \sin 2\theta = .75$ ✓



~~$2\theta = \sin^{-1}$~~

$\sin^{-1}(.75) = 48.6°$

$\therefore 2\theta = 48.6°$ or $180° - 48.6° = 131.4°$

$$\boxed{\begin{aligned} \theta &= 24.3°,\ 65.7° \\ \text{or} \quad &= .42,\ 1.14 \text{ radians} \end{aligned}}$$

10 a [4] $\sin(3u) = \sin u\,(3 - 4\sin^2 u)$

Let's convert LHS into an expression containing only $\sin u$ and see if it's equal to RHS.

~~sin~~ LHS $= \sin(2u + u)$ ✓

$= \sin 2u \cos u + \cos 2u \sin u$ ✓

$= (2\sin u \cos u)\cos u + (\cos^2 u - \sin^2 u)\sin u$ ✓

$= 2\sin u \cos^2 u + (1 - 2\sin^2 u)\sin u$

$= 2\sin u\,(1 - \sin^2 u) + (1 - 2\sin^2 u)\sin u.$

$= \cancel{\sin u\,(}$

$= 2\sin u - 2\sin^3 u + \sin u - 2\sin^3 u$

$= \sin u\,(3 - 4\sin^2 u)$

$=$ RHS

$\therefore$ It is an identity. ✓

10b [4] $-\ln|\sec\theta - \tan\theta| = \ln|\sec\theta + \tan\theta|$

$$\text{LHS} = \ln|\sec\theta - \tan\theta|^{-1}$$

$$= \ln\left|\frac{1}{\sec\theta - \tan\theta}\right|$$

$$= \ln\left|\frac{1}{\sec\theta - \tan\theta} \cdot \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right|$$

$$= \ln\left|\frac{\sec\theta + \tan\theta}{\sec^2\theta - \tan^2\theta}\right|$$

$$= \ln\left|\frac{\sec\theta + \tan\theta}{1}\right|$$

$$= \text{RHS}$$

$\therefore$ it is an identity.

10c [4] $\cos\left(\theta + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta)$

LHS $\cos\left(\theta + \frac{\pi}{4}\right) = \cos\theta \cos\frac{\pi}{4} - \sin\theta \sin\frac{\pi}{4}$

$= \frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta$

$= \frac{\sqrt{2}}{2}(\cos\theta - \sin\theta)$

$=$ RHS

it is an identity.

10d [4] $\csc\alpha = \sin\left(\frac{1}{\alpha}\right)$

at $\alpha = \frac{\pi}{2}$

LHS $= \csc\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = \frac{1}{1} = 1$

RHS $= \sin\left(\frac{1}{\pi/2}\right) = \sin\left(\frac{2}{\pi}\right) \approx .59$

LHS $\neq$ RHS

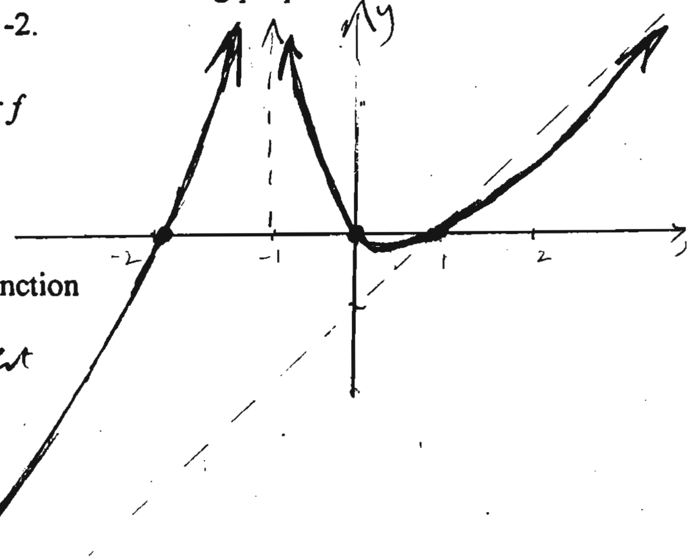
$\therefore$ it is not an identity.

I a) Sketch a graph of a function f that has all of the following properties.

(i) f has x-intercepts at 0,1 and -2.

(ii) as $x \to -1$, $f(x) \to \infty$

(iii) $y = x-1$ is an asymptote for f



b) Find a possible formula for the above function

$$f(x) = \frac{x(x-1)(x+2)}{(x+1)^2}$$ has the right intercepts and VA.

$= (x^3+x^2-2x)/(x^2+2x+1)$

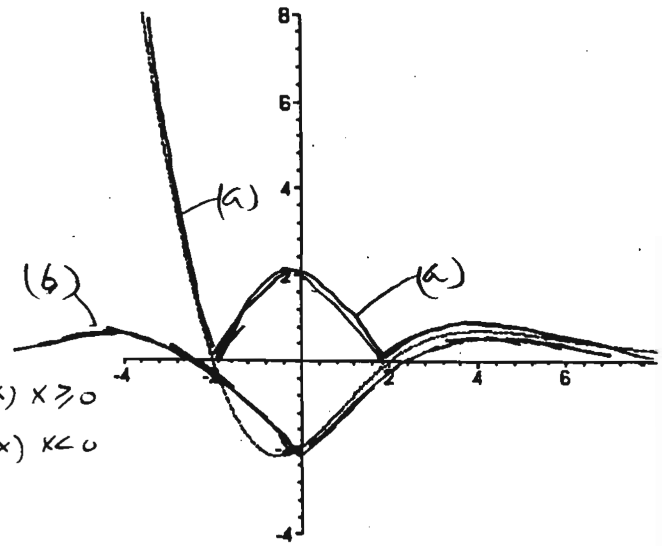
Checking the SA: –

$$\begin{array}{r} x - 1 \\ x^2+2x+1 \overline{) x^3 + x^2 - 2x} \\ \underline{x^3+2x^2+x} \\ -x^2-3x \\ \underline{-x^2-2x-1} \\ -x+1 \end{array}$$

← So $y = x-1$ is the SA

← Remainder

II The graph of a function f is shown here.



~~On the axes below~~ sketch graphs of

a) $g \circ f$ $\quad g \circ f(x) = g(f(x)) = |f(x)|$

b) $f \circ g$ $\quad f \circ g(x) = f(g(x)) = f(|x|) = \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases}$

where $g(x) = |x|$ is the absolute value function.