Gödel’s 1st incompleteness theorem (1931)

$\neg\exists x$ ProofOf(x, *g*)

Kurt Gödel’s achievement in modern logic is singular and monumental—indeed it is more than a monument, it is a landmark which will remain visible far in space and time. ... The subject of logic has certainly completely changed its nature and possibilities with Gödel’s achievement.

— John von Neumann

Gödel as a student (age 19)

Oskar Morgenstern on Gödel's citizenship interview

(Godel *massively* over-prepared for his citizenship hearing, studying US history and the Constitution in great detail)

Now came an interesting development. He rather excitedly told me that in looking at the Constitution, he had found some inner contradictions to his distress, and that he could show how in a perfectly legal manner it would be possible for somebody to become a dictator and set up a Fascist regime, never intended by those who drew up the Constitution. I told him that it was most unlikely that such events would ever occur, even assuming that he was right, which of course I doubted. But he was persistent and so we had many talks about this particular point. I tried to persuade him that he should avoid bringing up such matters at the examination before the court in Trenton, and I also told Einstein about it: he was horrified that such an idea had occurred to Gödel, and he also told him he should not worry about these things nor discuss that matter.

Gödel talks to the Examiner

good citizen. We assured him that this would certainly be the case, that he was a distinguished man, etc. And then he turned to Gödel and said, "Now, Mr. Godel, where do you come from?"

Gödel: "Where I come from? Austria."

The Examinor: "What kind of government did you have in Austria?"

Gödel: "It was a republic, but the constitution was such that it finally was changed into a dictatorship."

The Examinor: "Oh! This is very bad. This could not happen in this country."

Gödel: "Oh, yes, I can prove it."

The whole document is on albert.ias.edu (website for the Institute for Advanced Study) under "Oskar Morgenstern's account of Kurt Gödel's naturalization"

The Gödel sentence

- Gödel proved that **PA_1** (the theory generated by the Peano axioms in FOL) **is incomplete**, since there are true sentences of arithmetic that are not consequences of the first-order Peano axioms.
- He proved this by the simple means of writing down an FOL sentence of arithmetic, and showing it to be both true and unprovable.
- The Gödel sentence says, in effect, "I am unprovable"
  - Assuming the Peano axioms are true, and the rules of inference F+ are sound, **the Gödel sentence must be both true and unprovable**.

Add names and functions to FOL

Numerals: 0, s(0), s(s(0)), s(s(s(0))), etc.

Define +, $\times$ using s(x).

$m + 0 = m$

$m + s(n) = s(m + n)$

$m \times 1 = m$

$m \times s(n) = (m \times n) + m$

Peano axioms in FOL

A1. 0 is a natural number

A2. Every number *x* has a unique successor s(*x*).

A3. $\forall x\ s(x) \neq 0$

A4. $\forall x \forall y (s(x) = s(y) \rightarrow x=y)$

A5. The following is an axiom, for every wff with one free variable F(x):

$$[F(0) \wedge \forall y (F(y) \rightarrow F(s(y)))] \rightarrow \forall x\ F(x).$$

Two kinds of formal consequence

- We’re familiar with the “semantic” notion of first-order consequence (FO con).
- Let Φ be a set of FOL sentences, and P be an FOL sentence.

$\mathbf{\Phi \models P}$ means that P is a FO con of Φ.

- I.e. there’s no <interpretation, possible world> pair, where all the members of Φ are true but P is false.

$\mathbf{\Phi \vdash P}$ means that Φ proves P

- I.e. there is a formal proof with premises Φ and conclusion P.

Soundness and Completeness of $\mathcal{F}^+$

- Soundness of $\mathcal{F}^+$:
  - if $\Phi \vdash$ P then $\Phi \vDash$ P
- Completeness of $\mathcal{F}^+$:
  - if $\Phi \vDash$ P then $\Phi \vdash$ P
- N.B. $\mathcal{F}^+$ is both sound and complete

Theories and models

- A *theory* is a set of sentences that is closed under FO consequence.
  - I.e. it includes all the FO consequences of its members.
- A *model* of a theory T is an interpretation of the predicates, and a possible world, where all the members of T are true.
- A theory is *categorical* iff any two models of it are isomorphic.

Is the set of truths categorical?

- Let T_Ω be the *complete* theory of arithmetic in FOL.
- I.e. for every sentence P of first-order arithmetic, either $P \in \Omega$ or $\neg P \in \Omega$.
- I.e. T_Ω determines the truth value of every sentence of arithmetic.
- But T_Ω isn't categorical. It has non-standard models.

Is First-Order Peano Arithmetic Complete?

- No, as Gödel showed.
- There's a "diagonal" sentence that we can see is true, but which isn't a FO consequence of the Peano Axioms in FOL.

Gödel numbers of FOL symbols

(	)	0	s	$\neg$	$\vee$	$\wedge$	$\rightarrow$	$\leftrightarrow$	$\forall$	$\exists$	x	y	z	=
1	3	5	7	9	11	13	15	17	19	21	23	25	27	29

Gödel numbers of symbol strings

The Gödel number of a string of symbols (e.g. a wff or a sentence) is obtained by concatenating the Gödel numbers of each symbol making up the formula, putting a '0' after each number as a separator. (N.B. The Gödel number of Axiom 4 is 49 digits!)

E.g. Axiom 4: $\forall x \forall y(s(x) = s(y) \rightarrow x=y)$

$\forall$	x	$\forall$	y	(	s	(	x	)	=	s	(	y	)	$\rightarrow$	x	=	y	)
190	230	190	250	10	70	10	230	30	290	70	10	250	30	150	230	290	250	30

Gödel numbers of *lists* of strings

- To obtain the Gödel number of a **list** of formulas, write the Gödel numbers of the formulas in order, separating them by two consecutive zeros.

- Since a proof is just a sequence of FOL sentences, this method assigns a Gödel number to each proof.

Gödel "arithmetized" syntactic relations and properties

- An *arithmetical* property is a property of natural numbers, e.g. *prime* number, *even* number, number greater than 6, etc.
- Gödel showed that syntactic properties, like: *x is a sentence of FOL*, *x is a formal proof in F+ of y*, etc. can be reduced to arithmetical properties of the corresponding Gödel numbers, *and then expressed as wffs of FOL*.

1. "x *is divisible by* y": $y \mid x \Leftrightarrow \exists z \; x = y \times z$

2. $\text{IsPrime}(x) \Leftrightarrow \neg\exists z(z \neq 1 \wedge z \neq x \wedge z \mid x)$

8. $x \circ y$ = corresponds to the operation of concatenating" two finite sequences of numbers.

9. *seq*(x) corresponds to the number sequence that consists only of the number x

10. $paren(x) = seq(1) \circ x \circ seq(3)$

13. $not(x) \Leftrightarrow seq(9) \circ x$

14. $or(x, y) \Leftrightarrow paren(x) \circ seq(11) \circ paren(x)$

…

23. x *is a wff*

…

34. x *is a Peano axiom*

…

45. "x *is a proof of* y"   ProofOf(x, y)

(x is the Gödel number of a proof (in F+, using the Peano axioms as premises) of the sentence whose code number is y.)

“y *is provable*” : $\exists x$ ProofOf(x, y)

“y *is not provable*” : $\neg\exists x$ ProofOf(x, y)

The Diagonal Lemma

If P(x) is a wff with one free variable, then there is some natural number d such that d is the code number for P(d).

- But $\neg\exists$x ProofOf(x, y) is a wff with one free variable!
- Hence, there is some "diagonal" number *g* such that *g* is the code number of the sentence:

  $\neg\exists$x ProofOf(x, *g*).

- This sentence says, "I am not provable"