Gödel’s 1st incompleteness theorem (1931)

$\neg\exists x\ \mathrm{ProofOf}(x, g)$

Kurt Gödel’s achievement in modern logic is singular and monumental—indeed it is more than a monument, it is a landmark which will remain visible far in space and time. ... The subject of logic has certainly completely changed its nature and possibilities with Gödel’s achievement.

— John von Neumann

Gödel as a student (age 19)

The Gödel sentence

- Gödel proved that $\mathbf{PA_1}$ (the theory generated by the Peano axioms in FOL) **is incomplete**, since there are true sentences of arithmetic that are not consequences of the first-order Peano axioms.
- He proved this by the simple means of writing down an FOL sentence of arithmetic, and showing it to be both true and unprovable.
- The Gödel sentence says, in effect, “I am unprovable”
    - Assuming the Peano axioms are true, and the rules of inference F+ are sound, **the Gödel sentence must be both true and unprovable**.

Add names and functions to FOL

Numerals: 0, $s(0)$, $s(s(0))$, $s(s(s(0)))$, etc.

Define $+$, $\times$ using $s(x)$.

$m + 0 = m$

$m + s(n) = s(m + n)$

$m \times 1 = m$

$m \times s(n) = (m \times n) + m$

Peano axioms in FOL

A1. 0 is a natural number

A2. Every number *x* has a unique successor s(*x*).

A3. $\forall x\ s(x) \neq 0$

A4. $\forall x \forall y (s(x) = s(y) \rightarrow x=y)$

A5. $[F(0) \wedge \forall y(F(y) \rightarrow F(s(y)))] \rightarrow \forall x\ F(x)$.

Gödel numbers of FOL symbols

(	)	0	s	$\neg$	$\vee$	$\wedge$	$\rightarrow$	$\leftrightarrow$	$\forall$	$\exists$	x	y	z	=
1	3	5	7	9	11	13	15	17	19	21	23	25	27	29

Gödel numbers of symbol strings

The Gödel number of a string of symbols (e.g. a wff or a sentence) is obtained by concatenating the Gödel numbers of each symbol making up the formula, putting a '0' after each number as a separator. (N.B. The Gödel number of Axiom 4 is 49 digits!)

E.g. Axiom 4: $\forall x \forall y(s(x) = s(y) \rightarrow x=y)$

$\forall$	x	$\forall$	y	(	s	(	x	)	=	s	(	y	)	$\rightarrow$	x	=	y	)
190	230	190	250	10	70	10	230	30	290	70	10	250	30	150	230	290	250	30

Gödel numbers of *lists* of strings

- To obtain the Gödel number of a **list** of formulas, write the Gödel numbers of the formulas in order, separating them by two consecutive zeros.

- Since a proof is just a sequence of FOL sentences, this method assigns a Gödel number to each proof.

Gödel "arithmetized" syntactic relations and properties

- An *arithmetical* property is a property of natural numbers, e.g. *prime* number, *even* number, number greater than 6, etc.
- Gödel showed that syntactic properties, like: *x is a sentence of FOL*, *x is a formal proof in F+ of y*, etc. can be reduced to arithmetical properties of the corresponding Gödel numbers, *and then expressed as wffs of FOL*.

1. "x *is divisible by* y": $y \mid x \;\Leftrightarrow\; \exists z\ x = y \times z$

2. $\mathrm{IsPrime}(x) \Leftrightarrow \neg\exists z(z \neq 1 \wedge z \neq x \wedge z \mid x)$

8. $x \circ y$ = corresponds to the operation of concatenating" two finite sequences of numbers.

9. *seq*(x) corresponds to the number sequence that consists only of the number x

10. $\textit{paren}(x) = \textit{seq}(1) \circ x \circ \textit{seq}(3)$

13. $\textit{not}(x) \Leftrightarrow \textit{seq}(9) \circ x$

14. *or*(x, y) $\Leftrightarrow$ *paren*(x) $\circ$ *seq*(11) $\circ$ *paren*(x)

…

23. x *is a wff*

…

34. x *is a Peano axiom*

…

45. "x *is a proof of* y"   ProofOf(x, y)

(x is the Gödel number of a proof (in F+, using the Peano axioms as premises) of the sentence whose code number is y.)

“y *is provable*” : $\exists x$ ProofOf(x, y)

“y *is not provable*” : $\neg\exists x$ ProofOf(x, y)

The Diagonal Lemma

If P(x) is a wff with one free variable, then there is some natural number d such that d is the code number for P(d).

- But $\neg\exists$x ProofOf(x, y) is a wff with one free variable!
- Hence, there is some "diagonal" number *g* such that *g* is the code number of the sentence:

  $\neg\exists$x ProofOf(x, *g*).
- This sentence says, "I am not provable"