

# Truth and Possible Worlds

Richard Johns  
Langara College

We have defined logical entailment, consistency, and the connectives  $\wedge$ ,  $\vee$ ,  $\neg$ , all in terms of belief. In view of the close connection between belief and truth, described in the first section, we should see what each of these has to do with truth.

Suppose  $A$  is true, and  $A$  entails  $B$ . Does  $B$  then have to be true as well, or could it be false? If it is possible for  $B$  to be false here, then that's bad news. For then even a perfect thinker, starting with truth, could end up in error. Fortunately we shall see that this is not possible. Logical entailment cannot lead from truth to something false. Entailment, we say, is *truth preserving*.

## 1 Possible Worlds

There are many other links between the logical terms and truth. In order to investigate all of these it will be useful to introduce the notion of a *possible world*. This will allow to define the structure of propositions more precisely than before, and see just what it means for a proposition to be true.

Recall the idea of an *expansion* of an epistemic state – knowledge is added to the state, without removing any. This notion of expansion naturally leads to the idea of a *learning path* of epistemic states. A learning path is a sequence of epistemic states where each state is an expansion of the previous one. A learning path might be seen as the intellectual biography of a perfect thinker, as she acquires more and more knowledge.

One question about learning paths is whether they can ever come to an end. Is there any epistemic state that's so big it can't be expanded further? Theoretically, at least, a perfect thinker might get to the stage where she has a firm, definite opinion on every issue. She has unshakable beliefs about the doings of every single beetle in the Amazon basin, knows everything there is to know about every person on the planet, even the number of hairs on their head. Her epistemic state is every bit as detailed and complicated as the real world itself! No human could have such an epistemic state, of course; it's just a theoretical entity.

Such an epistemic state can be called *maximal*, since it cannot be expanded. It is not contained in any epistemic state (apart from itself of course). We will call such a maximal epistemic state a *possible world*.

**Definition**     A *possible world* is a maximal epistemic state, i.e. one which cannot be expanded. It is not contained in any other epistemic state.

There are many possible worlds, in this sense, as there are many possible combinations of beliefs. A possible world doesn't have to match the real world, it only has to be a possible state

of belief for a perfect thinker. It has to be coherent, make sense, and so on, but it doesn't have to be true.

The opposite of a maximal epistemic state (possible world) is a *minimal* epistemic state. This is a state with no information at all, so that it is contained in every epistemic state, even itself. It is clear that there can only be one minimal state. (Why?) Assuming it exists, we will call this state  $K_O$ , and refer to it as the *empty* epistemic state.

**Definition**  $K_O$ , the *empty* epistemic state, is the unique state that is contained in every epistemic state.

Even though  $K_O$  is the state of *maximum ignorance*, it is not entirely empty of all belief. It does contain some beliefs that every rational thinker must share, such as “ $1+1=2$ ”, “all green things are green”, “if it happens, it happens”, and other trivial claims. Such beliefs, known as *logical truths*, are present in every possible epistemic state, and are thus present in  $K_O$  as well.

Using the empty epistemic state, we can define *baseless propositions*. In general, a proposition makes background assumptions, without which it makes no sense. For example, “The Loch Ness Monster weighs at least 40 tons” assumes that there is a monster in Loch Ness, among other things. Now consider a proposition that makes no assumptions at all, so that it's meaningful even in  $K_O$ , the empty state. We will call such a proposition *baseless*.

**Definition** A *baseless* proposition is one whose base is  $K_O$ , the empty epistemic state.

If you like geometry, you can think of a baseless proposition as like a *position vector*, which starts from the origin  $O$  of the space.  $K_O$  can be thought of as the origin of logical space. All learning paths begin at  $K_O$ .

Consider a baseless proposition that takes you from  $K_O$  to a possible world  $K_W$ . This proposition (call it ‘ $W$ ’) contains all the information that is in  $K_W$ , so we shall call it a “possible world” as well.

If  $A$  is a baseless proposition, and  $W$  is a possible world, then  $A$  and  $W$  are either consistent or inconsistent. If  $A$  and  $W$  are consistent, then there must be some proposition that entails them both. But, since  $W$  is maximal, no proposition entails  $W$  except  $W$  itself. Therefore, if  $A$  is consistent with  $W$ , then  $W \Rightarrow A$ . If  $A$  is inconsistent with  $W$ , then  $W \Rightarrow \neg A$ , of course.<sup>1</sup> Thus, for any world  $W$  and proposition  $A$ ,  $W$  either entails  $A$  or it entails  $\neg A$ .

---

<sup>1</sup>We found out earlier that, if two propositions are inconsistent, then each entails the negation of the other.

## 2 Propositions Defined Properly

Let's start with some baseless proposition  $A$ . We have seen that, for every possible world  $W$ , either  $W$  entails  $A$  or it entails  $\neg A$ . Thus  $A$  partitions the class of possible worlds into two separate groups; one group of worlds that entail  $A$ , and another that entail  $\neg A$ . Let us call the first group the " $A$ -worlds". You might think of these as the worlds according to which  $A$  is true. According to each of the other worlds,  $A$  is false.

Perhaps the fundamental fact about propositions is this:

The (baseless) proposition  $A$  is the disjunction of all the  $A$ -worlds<sup>2</sup>

The proposition  $A$ , in other words, is the information that is in common between all the  $A$ -worlds; it is the point of strongest agreement. This fact enables us to investigate the relation between truth and belief.

We said before that, when someone utters a sentence, they are trying to cause a change in the listener's epistemic state. We can now see more clearly what that change is. Suppose Fred tells Betty: "Your cat has shredded my logic assignment". Let the proposition expressed by this sentence be  $A$ . Now consider the minimal state where  $A$  is fully believed, which we will call  $K_A$ . This is actually the strongest epistemic state that is weaker than all the  $A$ -worlds. Since Fred wants Betty to believe  $A$ , Fred wants Betty's epistemic state to be stronger than  $K_A$ .

Fred doesn't want Betty to lose any of her current knowledge, so her new epistemic state should be an expansion of her old one,  $K_{old}$ . So Betty's new state,  $K_{new}$ , should be an expansion of  $K_A$ , and also an expansion of  $K_{old}$ . Moreover, since Fred isn't saying anything apart from  $A$ ,  $K_{new}$  should be the weakest such state. This state we will call  $(K_{old} \wedge K_A)$ , since it is the weakest state that is stronger than both  $K_{old}$  and  $K_A$ . Thus:

The proposition  $A$  is associated with the expansion from  $K$  to  $(K \wedge K_A)$ .

In other words, coming to believe the proposition  $A$  means that your epistemic state moves up to the closest state in which  $A$  is believed.

## 3 Truth

The possible worlds are all inconsistent with each other. This follows from the definition of consistency. Since a possible world is only entailed by itself, there cannot be any proposition that entails two distinct possible worlds. Therefore, no two possible worlds can be consistent.

---

<sup>2</sup>In a more rigorous treatment of the subject, possible worlds are defined first, and then propositions are defined in terms of them. Such a treatment would be harder to understand, however.

(Any two completely opinionated people will disagree, unless they *completely* agree.) Note however that two worlds can agree on some things; they just can't agree on everything.

We shall assume that precisely one of the possible worlds is true. (By 'true', I mean that it exactly matches the real world, the world that actually exists. One of the possible worlds is like a perfectly detailed map that corresponds exactly to the actual territory.) This assumption that there is one true world can be split into two parts: (i) at least one possible world is true, and (ii) no more than one possible world is true.

The first assumption says that thought is capable, at least, of capturing reality. It would be rather depressing if every possible world were false. The second assumption is grounded in the fact that the possible worlds are mutually inconsistent, so that only one can be believed. If two or more of the worlds could be true, then belief would always be seriously different from truth, even under the best possible circumstances. This would also be depressing. Thus, when I assume that precisely one of the possible worlds is true, I am hoping for the best. Belief seems to have no purpose other than to represent truth, so I'm hoping that it can, in principle, succeed at this. I'm hoping that thought is not essentially dysfunctional. (This hope is bolstered by the thought that to say otherwise seems incoherent.)

This assumption that there is one true possible world is not granted by all. It is part of a philosophical view called *realism*, so those who deny it are usually called *anti-realists*. I don't require everyone here to be a realist, but you should understand what the view says and see what it's based on. Standard, classical logic assumes realism, so it makes sense to teach you about realism in this class. There are other logics that don't assume realism, such as intuitionistic logic and paraconsistent logic, which you can learn about in other philosophy classes.

So, for this class, let's assume that there is one true possible world. This proposition contains all the truth there is: the whole history of the universe, in perfect detail, all of science and math, the whole lot. It's like a "God's eye view" of reality. Let's call this world  $W_T$ , for "the Truth".

What is it for an ordinary proposition, one that is not a possible world, to be true? It simply has to be *part* of the truth. In other words,  $A$  is true just in case  $W_T \Rightarrow A$ , so that  $W_T$  is an  $A$ -world.

**Definition**     A proposition  $A$  is true just in case  $W_T \Rightarrow A$ .

We can now state the two most important theorems, that connect entailment and consistency to truth.

**Theorem**     If  $A$  is true, and  $A \Rightarrow B$ , then  $B$  is true as well. (Entailment preserves truth.)

*Proof:*     Suppose  $A$  is true, i.e.  $W_T \Rightarrow A$ . Then, since  $A \Rightarrow B$ , and entailment is transitive, it follows that  $W_T \Rightarrow B$ . In other words,  $B$  is true as well. ■

**Theorem** If  $A$  is inconsistent with  $B$ , then  $A$  and  $B$  are not both true. (True propositions are mutually consistent.)

*Proof:* Suppose that  $A$  and  $B$  are both true, i.e.  $W_T \Rightarrow A$  and  $W_T \Rightarrow B$ . Then, since some proposition entails both  $A$  and  $B$ , it follows that they are consistent. Thus, given that  $A$  is inconsistent with  $B$ , it follows that they are not both true. ■

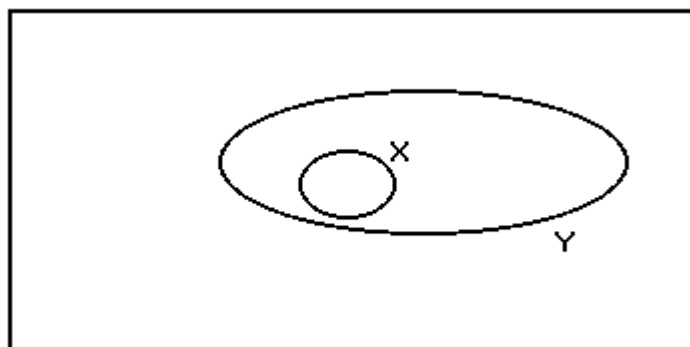
#### 4 Set Theory and Venn Diagrams

We have seen that every proposition  $A$  is associated with a class of worlds, known as the  $A$ -worlds – these are the possible worlds that each entail  $A$ . This fact enables us to represent the logical relations of entailment and consistency on a diagram, known as a Venn diagram.

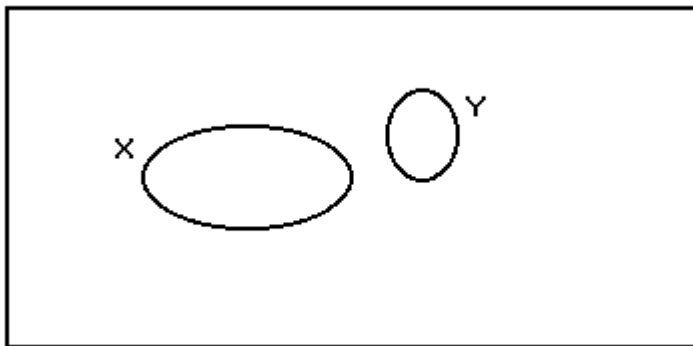
You've probably used Venn diagrams before, to represent relations between sets, or classes, and operations on classes. A set (or class) is a collection of objects, in no particular order. The objects in a set are known as its *members*. A set is sometimes written down by listing its members, separated by commas, between braces. For example,  $\{2, 4, 5\}$  is the set whose members are 2, 4 and 5. There is one set with no members, written  $\{\}$  or  $\emptyset$ . If two sets have the same members, then they are actually the same set. We assume that all objects are members of the *universal set*  $U$ .

Here are the basic relations:

1.  $X$  is a *subset* of  $Y$ , or  $X \subseteq Y$ . This means that every member of  $X$  is also a member of  $Y$ . This relation is shown on the following diagram. The universal set  $U$  is the whole rectangle.

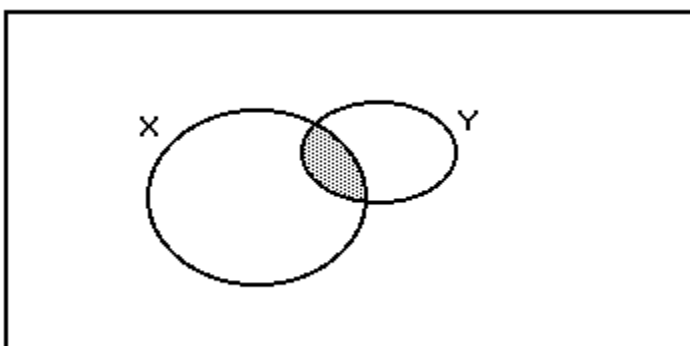


2.  $X$  and  $Y$  are *disjoint*. This means that they have no members in common. You can see, in the Venn diagram, that the loops don't overlap, so that no object can be inside both.

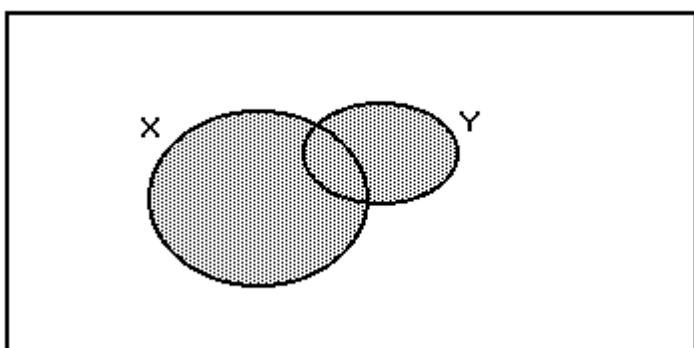


Here are the basic *operations* on sets. These are ways to form a new set out of one or two old sets.

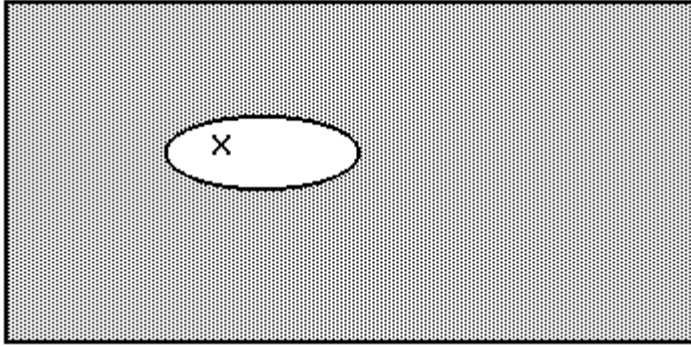
1. The *intersection* of X and Y. This is the set of all the objects that are in both X and Y. In the diagram, the shaded area is the intersection of X and Y, written  $X \cap Y$ .



2. The *union* of X and Y. This is the set of all the objects that are either in X, or in Y, or both. The shaded area on the next diagram is the union of X and Y, written  $X \cup Y$ .



3. The *complement* of X. This is the set of all the objects that are in the universal set, but not in X. It is the shaded area in the diagram below. The complement of X can be written  $X^c$ .




---

### Exercise 1

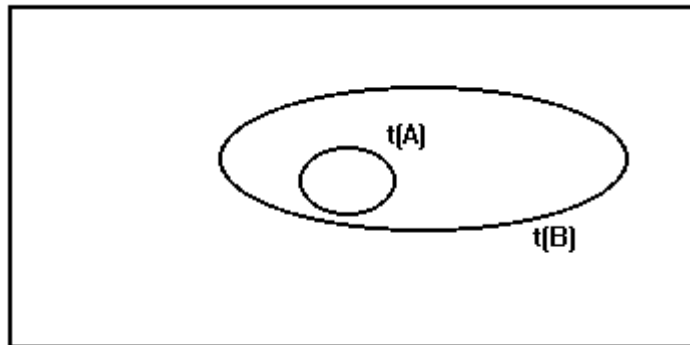
1. Let  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ ,  $X = \{2, 4, 6, 8, 10\}$ ,  $Y = \{6, 7, 8, 9\}$ ,  $Z = \{2, 4\}$ .
    - a. What is the relation between  $X$  and  $Z$ ?
    - b. What is the relation between  $Y$  and  $Z$ ?
    - c. What is the relation between  $X$  and  $Y$ ?
    - d. What is  $X \cap Y$ ? (Write down the elements, inside braces.)
    - e. What is  $X \cap Z$ ?
    - f. What is  $Y \cap Z$ ?
    - g. What is  $Y \cup Z$ ?
    - h. What is  $X \cup Y$ ?
    - i. What is  $X^c$ ?
    - j. What is  $Y^c$ ?
  2. On a Venn diagram, draw  $X$  and  $Y$  as intersecting sets, where neither is a subset of the other. Shade the set  $X \cap Y^c$ .
  3. On a similar diagram, shade the set  $(X \cup Y)^c$ .
- 

### 5 Entailment

Now we can get back to logic. You can draw a proposition on a Venn diagram, because each proposition is associated with a *set* of possible worlds. To show a proposition  $A$ , we can draw a loop around the set of  $A$ -worlds, i.e. the worlds that entail  $A$ . Let the set of all the  $A$ -worlds be written as  $t(A)$ , the *truth set* of  $A$ . Logical relations can then be shown as relations between sets. The rectangle now represents the set of all possible worlds.

1.  $A$  entails  $B$ . This holds just in case  $t(A) \subseteq t(B)$ . You might have expected the exact opposite here! (Many students do.) Since  $A$  is a stronger proposition than  $B$ , and contains more information, you might expect  $t(A)$  to be a bigger set than  $t(B)$ . In fact, as you see below, it's a

smaller set. You have to remember that each point on the diagram is a possible world, which is a very strong proposition.  $A$  is the *disjunction* of all the members of  $t(A)$ , i.e. the common content between all of them. Now, the more worlds there are in  $t(A)$ , the *less* there is in common between them all. So, the stronger the proposition  $A$  is, the *smaller* the set  $t(A)$ .



*Proof:*

1. Suppose  $A \Rightarrow B$ . For any  $A$ -world  $W$ ,  $W \Rightarrow A$ . Then, since  $A \Rightarrow B$ ,  $W \Rightarrow B$  as well. So any  $A$ -world is also a  $B$ -world, i.e.  $t(A) \subseteq t(B)$ .
2. Suppose  $t(A) \subseteq t(B)$ . It follows that every  $A$ -world entails  $B$ . Now,  $A$  is the disjunction of the  $A$ -worlds, so  $A$  is the strongest proposition that it entailed by every  $A$ -world. So  $A$  is stronger than  $B$ , i.e.  $A \Rightarrow B$ .

## Exercise 2

Consider the three propositions:

$A$  = "Janet is 5' 8" tall"

$B$  = "Janet is between 5' 5" and 5' 9" tall"

$C$  = "Janet is between 5' and 6' tall"

1. Which proposition entails the other two? (i.e. which is the strongest?)
2. Which proposition is the weakest?
3. Which proposition has the widest range of possible heights for Janet?
4. Which proposition has the smallest range of possible heights for Janet?

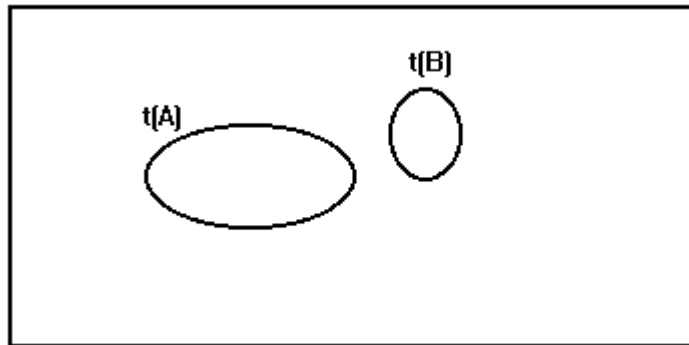
## 6 Consistency

Propositions  $A$  and  $B$  are consistent just in case there is some proposition that entails them both. Suppose that proposition is  $C$ . Then each  $C$ -world will entail both  $A$  and  $B$ , obviously, and so each  $C$ -world is both an  $A$ -world and a  $B$ -world. Thus the set of  $A$ -worlds intersects the set of  $B$ -

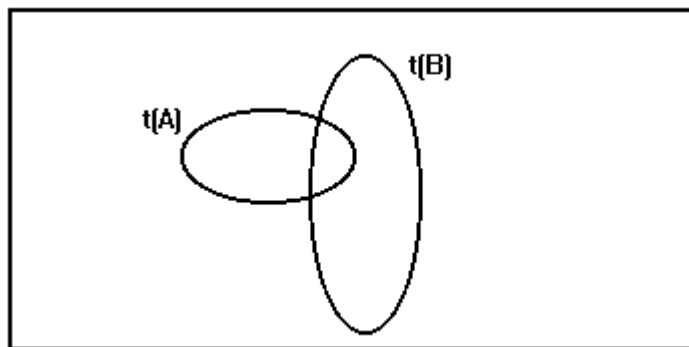


worlds. If there is no such intersection, then it is easy to see that the propositions are inconsistent. We can show this on Venn diagrams.

First, this is what inconsistent propositions look like:



The sets of worlds, as you can see, are disjoint. Consistent propositions, on the other hand, have a non-empty intersection, like this:



## 7 Truth and the Connectives

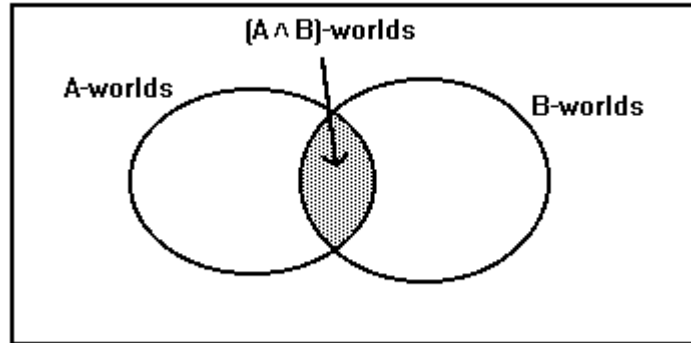
$\wedge$

Suppose  $A$  is true, and  $B$  is also true. Doesn't it follow that the conjunction  $A \wedge B$  is true as well? Yes it does, and we can now show this. In the work that follows it will be helpful to say “ $A$  is true in  $W$ ” when  $W \Rightarrow A$ , and “ $A$  is false in  $W$ ” when  $W \Rightarrow \neg A$ . These terms are a little misleading, as truth is really a matter of correspondence to the real world. What is really meant by “ $A$  is true in  $W$ ” is that  $W$  claims that  $A$  is true.

**Theorem**      $A \wedge B$  is true in all and only the worlds where  $A$  and  $B$  are both true.  
 I.e.  $t(A \wedge B) = t(A) \cap t(B)$ .

*Proof:*     This is most easily seen using a Venn diagram.  $A \wedge B$  is the weakest proposition that is entailed by  $A$ , and entailed by  $B$ . Thus  $t(A \wedge B)$  must be the *largest* set of worlds that is

a subset of  $t(A)$ , and also a subset of  $t(B)$ . This set is, of course, the intersection of  $t(A)$  with  $t(B)$ . ■



The shaded area on this diagram is the class of  $(A \wedge B)$ -worlds. We can see that this is the intersection of the  $A$ -worlds with the  $B$ -worlds.

We see that there are basically four kinds of world in this diagram. There are worlds where (i)  $A$  and  $B$  are both true, (ii)  $A$  is true and  $B$  is false, (iii)  $A$  is false and  $B$  is true, and (iv)  $A$  is false and  $B$  is false. The proposition  $A \wedge B$  is true only in worlds of the first kind.

In view of this fact, the information in this diagram can also be presented in a table, known as a *truth table*, as follows:

$A$	$B$	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

The symbol 'T' means true (in those worlds), and 'F' means false (in those worlds).

If you look at the two columns on the left, under ' $A$ ' and ' $B$ ', you will see that there is one row for each of the four kinds of world. The column on the right, under ' $A \wedge B$ ', says whether  $A \wedge B$  is true or false in each of these kinds of world.

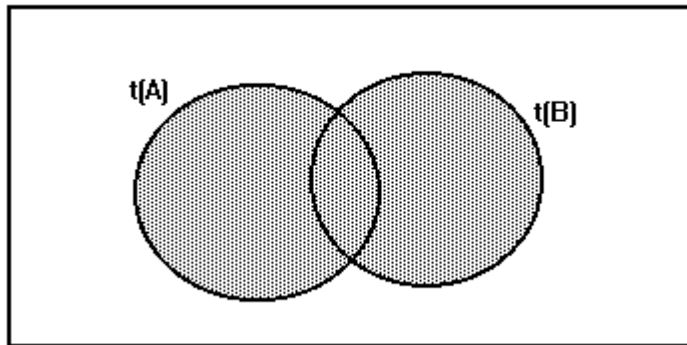
Note that the rows of a truth table are always placed in this order, from top to bottom: TT, TF, FT, FF. This is an arbitrary choice, but if we all do it the same way then your work will be much easier to mark!

Truth tables are much more commonly used than Venn diagrams. Partly this is because they can handle more than three propositions at once, unlike Venn diagrams. They are also neat and compact.

If a sentential connective has a complete truth table, then it's called *truth functional*. This means that the truth value of the “output” proposition is determined by the truth values of the “input” propositions. (In this case, the inputs are  $A$  and  $B$ , and the output is  $A \& B$ .)

✓

We can look at disjunction ( $\vee$ ) in the same way. It turns out that the class of  $(A \vee B)$ -worlds, i.e.  $t(A \vee B)$ , is the *union* of  $t(A)$  with  $t(B)$ . This may be shown on a Venn diagram:



The shaded area shows the class of  $(A \vee B)$ -worlds, i.e.  $t(A \vee B)$ . We may also use a truth table:

$A$	$B$	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

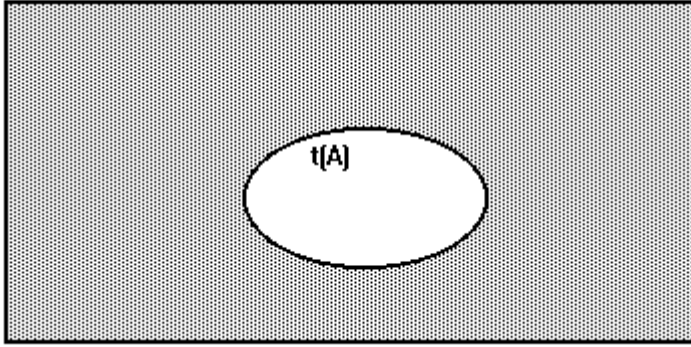
Note that the two columns on the left of this table are the same as for ‘ $\&$ ’. In the right-hand column, under ‘ $A \vee B$ ’, note that there are more ‘T’s than before.  $A \vee B$  is true if either  $A$  is true,  $B$  is true, or both. It’s only false when  $A$  and  $B$  are both false.

I haven’t shown you an argument that this table is correct, but it’s fairly obvious. Since  $A \vee B$  is the strongest proposition entailed by both  $A$  and  $B$ ,  $t(A \vee B)$  must be the smallest set of which  $t(A)$  and  $t(B)$  are both subsets. This set is just  $t(A) \cup t(B)$ .

Note that  $\vee$ , like  $\&$ , is a truth-functional connective.

¬

The negation of  $A$ , you will recall, is the weakest proposition that is inconsistent with  $A$ . Therefore,  $t(\neg A)$  is the largest set that is disjoint with  $A$ . This set is just the complement of  $t(A)$ . Thus  $t(\neg A)$  is the shaded area on the diagram below.



The truth table for negation is then as follows.

$A$	$\neg A$
T	F
F	T

You can see that  $A$  and  $\neg A$  have opposite truth values, in every world.

---

### Exercise 3

1. For each pair of propositions below, use a truth table to show that the propositions are identical, i.e. they are true in exactly the same class of possible worlds.

- $A \wedge B, B \wedge A$ .
- $A, \neg \neg A$ .
- $A, A \wedge A$ .
- $G, G \vee G$ .
- $\neg(A \vee B), \neg A \wedge \neg B$ .
- $\neg(A \wedge B), \neg A \vee \neg B$ .
- $A \wedge (B \vee C), (A \wedge B) \vee (A \wedge C)$ .
- $A \vee (B \wedge C), (A \vee B) \wedge (A \vee C)$ .
- $(A \wedge B) \vee C, (A \vee B) \wedge C$

2. Show the following logical relations on Venn diagrams. (For each part, draw a separate diagram in which the claim(s) hold(s). Note that there should be no shading.) Make your diagram as general as possible; i.e. it should not show any logical relation that has not been asked for.

- $B \Rightarrow \neg A$
- $B \Rightarrow A, C \Rightarrow A$ .
- $(B \wedge C) \Rightarrow A$

d.  $B \Rightarrow C, C \Rightarrow A$

e.  $(B \vee C) \Rightarrow A$

f.  $A \Rightarrow (B \wedge C)$

g.  $A \Rightarrow B, A \Rightarrow C.$