Probability and Symmetry

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Abstract: The Principle of Indifference, which dictates that we ought to assign two outcomes equal probability in the absence of known reasons to do otherwise, is vulnerable to well-known objections. Nevertheless, the appeal of the principle, and of symmetry-based assignments of equal probability, persists. We show that, relative to a given class of symmetries satisfying certain properties, we are justified in calling certain outcomes equally probable, and more generally, in defining what we call relative probabilities. Relative probabilities are useful in providing a generalized approach to conditionalization. The technique is illustrated by application to simple examples.

Word Count: 4,948

1. Introduction. A recent statement of the classical Principle of Indifference (PI) runs as follows:

> If there are n mutually exclusive possibilities h_1, …, h_n, and e gives no more reason to believe any one of these more likely to be true than any other, then $P(h_i / e)$ is the same for all i.[1]

In the absence of any known reason to assign two outcomes different probabilities, they ought to be assigned the same probability.

The Principle is generally attributed to Laplace, who spoke of cases as "equally possible" for us when there is "nothing to make us believe that one case should occur rather than any other."[2] One objective of this paper is to make sense of the quaint notion of "equi-possibility" as distinct from equal probability.[3] The Laplacian formulation is of more than historical interest.

PI is most frequently applied on the basis of an appeal to symmetries among the possible outcomes. A coin of unknown origin appears symmetrical, so we should assign equal probability to either face's coming up when the coin is tossed. A six-sided die appears symmetrical, so each face has probability 1 in 6 of landing uppermost.

Symmetry-based applications raise two well-known types of concern. First, there is an epistemological worry: the relevant symmetries between possible outcomes may be merely apparent. The die might actually be weighted so that 6 comes up with disproportionate

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[1] Howson and Urbach [1993], 52. The Principle of Indifference is also commonly referred to as the Principle of Insufficient Reason.

[2] See Laplace [1902], VII, 181.

[3] See Hacking [1975] for a discussion of Laplace's definition.

frequency. The concern here is that our epistemic probability for a certain event might not agree with its actual physical probability, or chance. In this paper, that concern will not be relevant because we shall not address the issue of applications of PI to physical probability.[4] We confine our discussion to epistemic probability, because symmetry arguments occur most naturally in that context. We do not usually talk about real physical symmetries, but symmetries in some model or representation of the space of possible outcomes.

Second, there is a logical worry: different ways of describing the outcome space might yield different classes of symmetries that generate inconsistent probability assignments. Notoriously, this situation occurs in the Bertrand Paradox, in which three distinct symmetry-based arguments can be given to support three different answers to the question: what is the probability that a needle dropped on a unit circle will determine a chord of length greater than or equal to $\sqrt{3}$?[5]

Despite the paradoxes, the appeal of symmetry arguments and the Principle of Indifference persists. Such arguments are of great importance in assigning prior probabilities. Although we do not directly confront the problem of paradoxes in this paper, we believe that they can be avoided by a careful formulation of epistemic symmetry. In this paper our aim is more modest: we take a suitable set of symmetries as given, and impose conditions that ensure that no conflicting probability assignments can arise in the examples we discuss. In general, we do not endorse the idea that mere ignorance justifies the application of PI. Symmetry between possible outcomes, within a state of knowledge, is a very special case of ignorance, and it is only in such cases that PI can be applied.

These symmetries sometimes define not just a relation of equi-possibility, but also a generalization to what we call *relative probabilities*. Relative probabilities provide a natural definition of conditional probability that agrees with the ordinary one in normal cases, but also extends to cases where symmetry arguments appear to justify conditionalizing on outcomes that have probability zero or, indeed, have no well-defined probability. The technique has two main advantages over other approaches to this type of conditionalization: it is mathematically quite simple, and it involves no arbitrary assumptions.

2. Examples. To support our view about the importance of symmetry arguments and PI, consider the following examples:

Example 2.1 (The dartboard):

> You are throwing a dart at a square board that measures one meter by one meter. Suppose that the point of the dart is equally likely to hit any point on the board. Since the entire board has unit area, the probability that the dart will land in a specified region is just its area, if that area is well-defined. In particular, any two regions of equal area are equally likely to contain the point where the dart hits.

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[4]For a recent discussion, see Strevens [1998].
[5] Bertrand [1889].
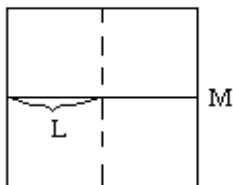
Now consider the proposition *M* that the dart lands on a given horizontal line crossing the dartboard, as shown. There is zero probability that a dart will land exactly along this line: $Pr(M) = 0$. But nothing prevents us from raising the question: what is the conditional probability of *L*, the dart's landing on the left half of *M*, given that it lands on *M*? Intuitively, and by appealing to the symmetry of the left and right halves, the answer is ½. Yet $Pr(L\,/\,M)$ cannot be obtained from the usual definition of conditional probability, $Pr(L\,/\,M) = Pr(L \cdot M)\,/\,Pr(M)$, since the denominator is 0.[6]

Example 2.2 (Uniform distribution on the plane):

Suppose the dartboard is now the entire plane. That is, we want a way to select a point in the plane at random, or a uniform probability distribution over the entire plane. We want to say, for instance, that the probability that the point lies above the *x*-axis is ½, or that with probability 1/4 it lies in the quadrant bounded by the positive x- and y-axis. Any infinite horizontal or vertical strip of width 1 should have an equal probability of containing the point.

No conventional countably additive probability distribution has these features. A probability function *Pr* is countably additive if whenever E_1, E_2, … are exclusive outcomes and each has a well-defined probability, then $Pr(\vee E_i) = \Sigma Pr(E_i)$, where the disjunction and sum run from 1 to ∞. Now the plane is a countable disjoint union of strips of width 1, namely those of the form k $\leq x <$ k+1 where k is an integer. There is no way to assign an equal positive value to the probability of each such strip, since the countable sum would then be infinite, while the probability that the point lies somewhere in the entire plane is just 1. Yet it does not seem unreasonable that there could be a uniform probability distribution which makes it equally likely that a point lies in any strip of width 1.

Example 2.3 (The de Finetti lottery):

Famously, de Finetti argued that we should be able to make sense of a lottery in which the number of tickets issued is countably infinite, one for each positive integer, and each ticket has an equal chance to win.[7] The claim amounts to the suggestion that we can have a uniform probability distribution over the natural numbers. A little reflection shows that the same difficulty arises as for the uniform distribution over the plane: there is no positive value that we can assign to each number in such a way that the countable

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[6] This example is discussed in Bartha and Hitchcock [1999], which provides a different analysis.

[7] De Finetti [1975], 123.

sum is 1. Conventional mathematics compels us to assign lower probability values to higher ticket numbers: if p_n is the probability that ticket number *n* wins, then $p_n \rightarrow 0$ as $n \rightarrow \infty$. Considerations of symmetry, however, make such a lottery seem reasonable. At the very least, we should not rule it out *a priori* simply because we lack a mathematical model.

De Finetti's conclusion, based on the lottery of Example 2.3, was that countable additivity should be dropped, and indeed, non-standard solutions to the De Finetti lottery introduce probability functions that are not countably additive. We have a different proposal. For certain problems, a class of symmetries rather than an absolute probability measure should be taken as the most fundamental notion. In each of the examples, symmetries appear to justify assignments of equal likelihood or, as we shall say, equi-possibility, to certain outcomes, as well as certain conditional probability assignments. The dart is *as likely* to hit the left half of the dartboard as the right half; one ticket is *as likely as* any other to win. Ordinary measures on these spaces do not allow us to derive the relevant conditional probabilities, because they would require us to conditionalize upon an outcome whose probability is zero.

Two techniques are commonly employed for dealing with such situations.[8] The first is to use Popper functions, which amounts to taking a conditional probability function C(A, B) as primitive. We can then assign a value C(A, B) to the conditional probability of A given B even when *Pr(B) = 0*. The absolute probability *Pr(B)* is defined as C(B, T).

In one sense, Popper functions are just what we need. For instance, in Example 2.1, we could just set C(L, M) = ½: problem solved! Our main objection here is that such an assignment appears arbitrary unless its dependence on considerations of symmetry is made explicit. The symmetries on the outcome space, not the Popper function, should be primitive.

The second approach is to use non-standard probability assignments. Roughly, the idea is that if B is a proposition that is not impossible, but would normally be assigned probability 0, we instead assign it an infinitesimal probability value. Then the conditional probability *Pr(A / B)* may still be defined by the usual ratio *Pr(A*⋅*B)/Pr(B)*. In Example 1, for instance, we would set *Pr(M)* = η for some positive infinitesimal η, and *Pr(L) = Pr(L ×M) =* η/2. Then *Pr(L / M) = Pr(L ×M) / Pr(M) =* ½, as required.

There are a number of concerns here. One problem is the arbitrariness in the choice of the infinitesimal η, which is not dictated by the problem. Of course, η cancels out in computing the conditional probabilities of interest, but what happens if we want to conditionalize yet again? For instance, suppose we further restrict our attention to a subset *E* of *M* consisting of just three points. We can argue, again by symmetry, that given that the dart lands in *E*, it is equally likely to be at any of the three points. This requires an assignment to *Pr(E)* of an infinitesimal an order of magnitude smaller than η. The non-standard approach appears to require a degree of mathematical complexity that far outstrips the much simpler appeal to symmetry that, in any case, really grounds our beliefs about the relevant conditional probabilities. More generally, the introduction of non-standard probabilities is unwarranted if a simpler approach can be developed.

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[8] See McGee [1994] for a presentation of both approaches and an argument that they are equivalent.

In this paper, we outline such an alternative, based on the principle that two symmetrical outcomes are equi-possible. Given a class of symmetries satisfying certain reasonable properties, we show that we can, in a relatively straightforward manner, define equi-possible sets and conditional probabilities.[9] In some cases, we can define a relation of equi-possibility (and relative probability) even though no corresponding probability measure is definable, which explains our opening gloss on Laplace's version of PI. There really can be relations of equi-possibility which precede any absolute probability assignment. In cases where probabilities are definable, however, the picture to keep in mind is the following:

Class of symmetries on X

$$\swarrow \qquad\qquad\qquad\qquad \searrow$$

$$Pr_X \qquad\qquad\qquad\qquad Pr_E$$
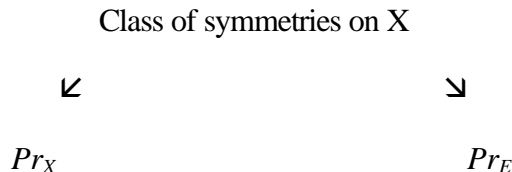
In the applications of interest, E is a subset of X that has probability 0. Pr_X is a probability distribution on the whole set of outcomes X. Pr_E is probability conditionalized upon E; we can think of it as the probability function that 'lives' on the tiny subspace of outcomes consistent with E. In general, there is no conventional way to pass from Pr_X to Pr_E (or vice versa), yet there is often an intuitive link, such as the idea that both distributions should be uniform. Our proposal is that we should regard the two probability functions as linked by a 'common cause', an underlying class of symmetries.

Section 3 introduces the concept of relative probability. Section 4 shows how symmetries can be used to define a relation of *commensurability* between sets. Section 5 shows how to define the relative probability of two commensurable sets. Finally, in section 6, the approach is applied to the three examples.

3. Relative probability. In this section, we provide a general, 'top-down' explication of relative probability, together with some basic axioms that the relation should satisfy. The objective is to show that the concept of the relative epistemic probability of two outcomes can be defined using a Ramsey-style approach. The definition plays no role, however, in what follows. In the remainder of the paper, a 'bottom-up' approach is developed from the starting point that if two outcomes are symmetric, they are equi-possible, i.e., they have relative probability 1. We examine what constraints are needed on the symmetries so that a relative probability function can be constructed.[10]

Suppose that $Pr(A) = k \cdot Pr(B)$. Then the expected monetary value of a contract that pays \$1 if A is k times the expected monetary value of a contract that pays \$1 if B:

$$val[\$1 \text{ if } A] = k \cdot val[\$1 \text{ if B}].$$

Equivalently,

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[9] We emphasize a few simple applications, since we believe that the approach developed here needs refinement before it can be generalized.

[10] Confession: in the following sections, we develop only a limited concept of relative probability which is not guaranteed to satisfy the additivity axiom (R3). We hope eventually to provide a fuller discussion.

val[\$1 if *A*] = *val*[\$k if B].

The idea of inverting this reasoning and defining subjective probability in terms of value or desirability is familiar.[11] The variation we wish to introduce here is to notice that sometimes we can make sense of the idea that two contracts have equal value even though neither contract, taken in isolation, has any value that makes sense! For example, it makes no sense to assign a precise value to a ticket in the de Finetti lottery: any positive value is too high, while zero is too low. Nevertheless, we can acknowledge that any two tickets have equal value, or that having two tickets is twice as valuable as having one. Accordingly, we generalize the relation above by writing

> [\$1 if *A*] ~ [\$k if B]

to signify that these two contracts are equally valuable. We can define relative probability in terms of this relation.

Definition 3.1 (Relative probability):

> R(A, B) = k iff [\$1 if *A*] ~ [\$k if B]; we say that k is the probability of *A relative* to *B*.

In the case where *A* and *B* both have a well-defined absolute probability and *Pr*(B) is non-zero, R(A, B) is just the ratio *Pr*(A) / *Pr*(B). In general, relative probability should satisfy the following axioms, which parallel the standard axioms for probability functions, and are all easily derived in cases where R(A, B) = *Pr*(A) / *Pr*(B) with *Pr*(B) ≠ 0.

(R1)   R(A, B) ≥ 0

(R2)   R(A, A) = 1

(R3)   If A and B are mutually exclusive and R(A, C) and R(B, C) are defined, then
         R(A ∨ B, C) = R(A, C) + R(B, C)

(R4)   If R(A, C) and R(B, C) are defined and R(B, C) ≠ 0, then

> R(A, B) = R(A, C) / R(B, C)

Note that if R is defined via a ratio of absolute probabilities, then R(A · B, B) = *Pr*(A · B) / *Pr*(B) is just the conditional probability *Pr*(A / B). However, R(A · B, B) might be definable even if *Pr*(B) = 0, and is thus a natural candidate for a more general notion of conditional probability.

4. Commensurable Sets. We move from talk of outcomes to a set-theoretic approach, which is simpler and, of course, equivalent. Let X be a set. A *symmetry* on X is a function σ: X → X that is 1-1 and onto and, generally, satisfies certain other properties, which we leave

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[11] See Jeffrey [1983], Ramsey [1931].

unspecified. Let S be a given class of symmetries on X. Two subsets A and B of X are *equi-possible* if B = σ(A) for some symmetry σ ∈ S; in this case, we will speak of B as a *copy* of A.

The philosophical foundation for the inference from symmetry to equi-possibility is the assumption that a symmetry, in one's epistemic state, between A and B means that there are no resources within that epistemic state for "pointing to" or distinguishing one of the outcomes in the pair {A, B}. Roughly speaking, every known feature used to assign likelihood is the same for A and B. It follows, in the terminology of the previous section, that one must regard the contracts [$1 if A] and [$1 if B] as equally valuable, since one cannot even tell them apart. This assumption entails that only a distinguished class of mappings count as symmetries. For example, if X is a finite-dimensional Euclidean space, a symmetry mapping σ must be an isometry, or distance-preserving function. If two plane figures are related by an isometry, then knowledge of inter-point distances alone does not allow one to tell them apart. This fundamental assumption underlies everything that follows.

Our objective is to define a relation of commensurability, intended to capture the idea that the likelihood of two sets (or the value of the appropriate contracts) can be compared. Before we can even make sense of equi-possibility, however, we have to make sure that the class of symmetries is neither too small nor too big.

Definition 4.1 (regularity):

A class S of symmetries is *regular* if it has the following two properties:

(G) S is a group under function composition; and

(M) For no non-empty subset C of X and positive integers m>n are there symmetries $\theta_1, \ldots, \theta_m$ and $\Psi_1, \ldots, \Psi_n$ in S such that

$$\bigcup_{i=1}^{m} \boldsymbol{q}_i(C) \subseteq \bigcup_{j=1}^{n} \Psi_j(C),$$

where both unions are disjoint.

The reason for the condition (G) is that we want equi-possibility to be an equivalence relation:

- For all A, A and A are equi-possible (hence, the identity map is in S).
- If A and B are equi-possible, then so are B and A (hence if σ ∈ S, σ^{-1} ∈ S).
- If A, B and B, C are equi-possible, then A, C are equi-possible (hence S should be closed under function composition).

We need to make sure that S is at least big enough to satisfy the group axioms.

The reason for the strange-looking condition (M) is to avoid the following counter-intuitive situation, which can arise if we are too liberal about what counts as a symmetry.

Let A be the positive integers, A_1 the positive even integers and A_2 the positive odd integers. Let X be the set of all integers. Let the set S of symmetries be the class of all 1-1 onto functions σ:X → X. We can find σ_1 which maps A to A_1 and σ_2 which maps

A to A_2. So $A = A_1 \cup A_2$ where A_1 and A_2 are disjoint, yet each of A_1 and A_2 is equi-possible with A.

In such a situation, employing the approach developed in the next section, we would be forced to say that A is twice as likely as itself! The condition (M) rules out this possibility, as well as any case where m copies of A could be placed inside n copies of A even though $m > n$. The label (M) is appropriate because the condition is intimately related to the monotonicity property of probability measures. Henceforth, we shall assume that the set S of symmetries on X is regular, and we will speak of (X, S) as a *symmetry space*. Our contention is that regularity is the minimum requirement for defining a notion of equi-possibility.

Proposition: Suppose S is a commutative group, i.e., $\sigma\tau = \tau\sigma$ for all σ,τ in S. Then S is regular, i.e., the property (M) is satisfied.

Proof: Suppose, for a contradiction, that we have symmetries $\theta_1, \ldots, \theta_m$ and $\Psi_1, \ldots, \Psi_n$ in S with $m > n$ such that

$$\bigcup_{i=1}^{m} \theta_i(C) \subseteq \bigcup_{j=1}^{n} \Psi_j(C),$$

where both unions are disjoint. *C*hoose any point $x \in C$. For some j, $\Psi_j(C)$ contains $\theta_i(x)$ and $\theta_{i'}(x)$ for distinct i and í; without loss of generality, assume $\Psi_1(C)$ contains $\theta_1(x)$ and $\theta_2(x)$. Then $\Psi_1^{-1}(\theta_1(x)) \in C$ and $\Psi_1^{-1}(\theta_2(x)) \in C$. So we have $\theta_2(\Psi_1^{-1}(\theta_1(x))) \in \theta_2(C)$ and also $\theta_1(\Psi_1^{-1}(\theta_2(x))) \in \theta_1(C)$. But if all of these operators commute, then $\theta_2(\Psi_1^{-1}(\theta_1(x))) = \theta_1(\Psi_1^{-1}(\theta_2(x)))$, so that $\theta_2(C) \cap \theta_1(C)$ is non-empty, a contradiction.

Although symmetry groups are not generally commutative, they are commutative for the following examples.

Example 4.1: Let X be the set of all integers, and let S be the set of all mappings of the form $\sigma(x) = x + k$, where k is any integer. Then S is a commutative group, and hence regular. Any two singleton sets are equi-possible.

Example 4.2: Let X be the Euclidean plane, and let S be the set of all translations, or mappings of the form $\sigma(x) = x + e$, where e is a constant vector. Then S is a commutative group.

The following definition of commensurability is sufficient for our purposes.[12]

Definition 4.2 (Commensurability):

Two sets A and B are *commensurable* relative to S if:

(1) for some subset C of X (called a *divisor* of A and B), there are symmetries $\theta_1, \ldots, \theta_m$ and $\Psi_1, \ldots, \Psi_n$ in S (with m, n > 0) such that

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[12] The definition can be generalized. As it stands, the definition only handles cases of commensurable sets A and B where R(A, B) is positive and rational. This can be extended to handle cases where R(A, B) is irrational, zero or infinite.

$$A = \bigcup_{i=1}^{m} \boldsymbol{q}_i(C) \text{ and } B = \bigcup_{j=1}^{n} \Psi_j(C)$$

where both unions are disjoint; or

(2) for some subset C of X, the pairs A, C and C, B are commensurable.

The first clause describes the case where A is composed of *m* disjoint copies of C and B is composed of *n* disjoint copies. It includes the special sub-cases where B is a copy of A, and where B can be written as n disjoint copies of A. The second clause extends commensurability to the smallest transitive relation consistent with the first two clauses. Clearly, commensurability is an equivalence relation.

> *Example 4.1 (continued)*: By applying clause (2), we can show that any two finite sets are commensurable. The set E of even integers and the set O of odd integers are commensurable. No finite set is commensurable with any infinite set.

> *Example 4.2 (continued)*: Using the terminology of Example 2.2, any two infinite horizontal strips of width 1 are commensurable. So are any two strips of different widths, so long as the ratio of widths is a positive rational number.

> How are two commensurable sets related? Introducing the following relations helps:

- $\text{comm}_1(A, B)$ if A and B have a divisor

- $\text{comm}_{n+1}(A, B)$ if $\text{comm}_i(A, B)$ fails for $1 \leq i \leq n$, but there is a C such that $\text{comm}_n(A, C)$ and $\text{comm}_1(C, B)$

Each comm_n is a symmetric relation, and $\text{comm}_n(A, B)$ amounts to the assertion that there is a sequence of sets $A=A_1, A_2, \ldots, A_{n-1}, A_n=B$ such that A_i and A_{i+1} have a divisor (and that this does not happen for any integer smaller than n). Clearly, $\text{comm}_n(A, B)$ implies that A and B are commensurable, and an inductive argument using Definition 4.2 establishes that if A and B are commensurable, then $\text{comm}_n(A, B)$ for some n.

5. Relative Probability Revisited. Relative probability generalizes the notion of equi-possibility. Given a regular set of symmetries, we want to define relative probability in a way that parallels the definition of commensurability:

Definition 5.1 (Relative Probability):

> Suppose (X, S) is a symmetry space, and A, B are commensurable subsets of X.

> (1) If $A = \bigcup_{i=1}^{m} \boldsymbol{q}_i(C)$ and $B = \bigcup_{j=1}^{n} \Psi_j(C)$ for some C, with both unions disjoint, set R(A, B) = *m/n*.

(2) If $R(A, C)$ and $R(B, C)$ are defined and $R(B, C) \neq 0$, set $R(A, B) = R(A, C) / R(B, C)$.

If R is well-defined, then the axioms of section 3 are satisfied.[13]   The following properties of R are easy consequences of definition 5.1:

1. If $B = \sigma(A)$, then $R(A, B) = R(B, A) = 1$.

2. If $B = \bigcup_{i=1}^{n} s_i(A)$, then $R(B, A) = n$.

3. $R(A, B) = 1 / R(B, A)$ if $R(B, A) \neq 0$.[14]

4. If $R(A, C)$ and $R(B, C)$ are defined and $R(A, C) > 0$, then

$$R(B, A) = R(B, C) / R(A, C)$$

Of course, we have to show that R is in fact well-defined. Whether R is well-defined for all regular symmetry spaces is an open question, but we can show that R is well-defined for the examples we have been developing, where the group of symmetries is commutative.

Proposition: If the group of symmetries S is commutative, then R is well-defined.

Proof: First suppose that A and B have two divisors, C and D. Write $A = mC$ and $B = nC$ to indicate that A can be written as m copies of C and B as n copies of C, as in Definition 5.1. Suppose now that $A = m'D$ and $B = n'D$, where the relevant symmetries are as in the diagram. We have to show that $m/n = m'/n'$.
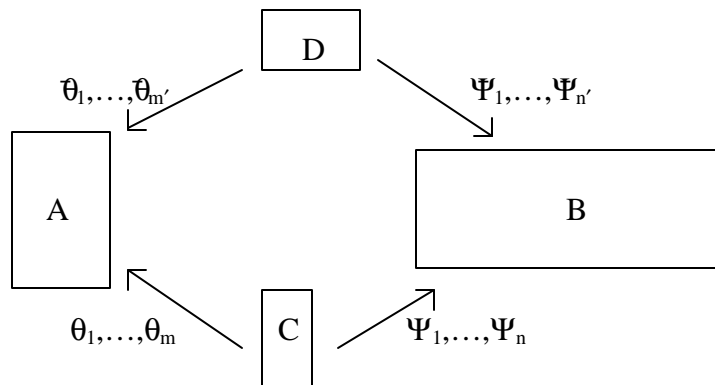
For each $c \in C$, and for $1 \leq i \leq m$, $\theta_i(c) = \theta_k(d)$ for some $d \in D$ and some k with $1 \leq k \leq m'$; also, for $1 \leq j \leq n'$, $\Psi_j(d) = \Psi_l(c^*)$, for some $c^* \in C$ and some l with $1 \leq l \leq n$. This shows that for any $c \in C$ and any subscripts i and j, there are $c^* \in C$ and subscripts k and l such that $\Psi_j(\theta_k^{-1}(\theta_i(c))) = \Psi_l(c^*)$; by commutativity, it follows that $\Psi_j \theta_i(c) = \theta_k \Psi_l(c^*)$. Thus,

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[13] Except possibly additivity; see note 10.

[14] Of course, as we have defined it, $R(B, A)$ is never 0. The restriction is mentioned here so that the properties will be valid even if the definition is extended to cover the case where $R(B, A) = 0$.

$$\bigcup_{i=1}^{m}\bigcup_{j=1}^{n'}\overline{\Psi}_{j}\boldsymbol{q}(C)\subseteq\bigcup_{k=1}^{m'}\bigcup_{l=1}^{n}\overline{\boldsymbol{q}}_{k}\Psi_{l}(C),$$

and analogous reasoning shows that containment holds in the reverse direction. By the (M) property, it follows that $m \cdot n' = m' \cdot n$.

For the general case, A and B are linked by intermediates C_1 and C_2, as in clause (2) of Definition 5.1. Following the discussion at the end of section 4, we can link each of A and B to each of C_1 and C_2 by a finite sequence of intermediate sets, such that each pair in succession has a divisor. The resulting diagram is a generalized version of the one above, with two sets of zig-zag patterns joining A to B, one via C_1 and the other via C_2. By picking one of the divisors and constructing the same sort of argument as just given, commutativity and the (M) property establish that $R(A, C_1) / R(B, C_1) = R(A, C_2) / R(B, C_2)$.

6. Applications. Summarizing, we have shown that if the set of specified symmetries on a space of outcomes X is a commutative group, then the relative probability R(A, B) of any two commensurable sets A and B is well-defined by Definition 5.1. The restriction to commutative groups of symmetries is merely a sufficient condition that may be removed in a more complete study of relative probability than we can provide here. Despite this limitation, we are now able to deal, at least in a preliminary manner, with the symmetry arguments presented in the three examples of section 2.

Examples 2.1 and 2.2: Take X to be the plane and S to be the set of all translations $\sigma(x) = x + e$, for a fixed vector e. S is a commutative group, so relative probabilities are well-defined. Of course, we should properly include other isometries such as rotations and reflections in S, not just translations, but then commutativity would fail. The restriction to translations means that we cannot regard A and $\theta(A)$ as equi-possible, where θ is a non-trivial rotation or a reflection. A fuller treatment of relative probabilities should eliminate this restriction,[15] but we are nevertheless able to justify some of the claims about relative probability mentioned in section 2.

Taking S to include all translations implies that two regions related by a translation are indistinguishable from a probabilistic point of view. In Example 2.2, for instance, any two horizontal strips of equal width are clearly equi-possible, or have relative probability 1, since they are related by a translation. Two strips having widths whose ratio is a positive rational number *r* have relative probability *r*, since we can find a common divisor.

In Example 2.1 (the dartboard), we simply restrict the relative probability function to pairs of sets that are subsets of the unit square, D. It is then obvious that the horizontal line *M* can be written as two copies of *L*, so that $R(L, M) = ½$.

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[15] Actually, there might be two ways to develop a more adequate notion of relative probability in this setting. One is to expand the set of symmetries S as suggested, including all isometries. Another is to extend the relation of commensurability and the relative probability function R to countable unions of sets. Translation invariance is enough to establish the uniqueness of Lebesgue measure as the only countably additive measure (up to a constant multiple — see (Rudin 1974)); an analogous result might be possible for a relative probability function.

Example 2.3: Take X to be the set of all integers and S to be the set of translations by an integer, $\sigma(x) = x + k$. Again, S is commutative, so relative probabilities are well-defined. If we restrict our attention to the relative probability of pairs of sets that are subsets of the positive integers, we get the uniform distribution that de Finetti wanted. Any two singletons are obviously equi-possible. So are the sets of even and odd numbers.

As noted in section 2, one advantage of approaching conditionalization using relative probabilities is the possibility of repeated conditionalization on outcomes with zero probability. In example 2.1, we might first learn that the dart landed on a particular horizontal line; we then learn that it landed at a point on that line with rational co-ordinates; finally we learn that it landed at one of three points. In each case, we can talk about relative probabilities, based on symmetries.

The definitions above are tentative, and the idea of making symmetries the fundamental notion in a probability assignment is still far from vindicated. Nevertheless, the above discussion should contribute to defending the viability of Laplacian "equi-possibility" as a notion that can stand independently of probability.

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