Philosophy 1102: Introduction to Logic

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Proof Strategies

1. *Only* make an assumption for one of six reasons, as follows:

- (i) You're trying to prove $\neg$P by reductio. (Assume P, and get $\perp$.)
- (ii) You're doing a proof by cases, using a known disjunction, say P $\vee$ Q. (You assume P, derive the goal, then discharge that and assume Q, then derive the goal again.)
- (iii) You're trying to prove a conditional P $\rightarrow$ Q. (Assume P and derive Q.)
- (iv) You're trying to prove a simple universal sentence, $\forall$xP(x). (The assumption line only has a constant, such as a, in a box, and nothing else. The last line of the subproof is P(a).)
- (v) You're trying to prove a universal conditional sentence, $\forall$x(P(x) $\rightarrow$ Q(x)). (The assumption line has a constant, such as a, in a box, and P(a). The last line of the subproof is Q(a).)
- (vi) You're planning on eliminating an existential sentence $\exists$xP(x). (The assumption line has a constant, such as a, in a box, as well as the sentence P(a). The last line of the subproof is the goal, which must not contain the constant a.)

Never make an assumption for other reasons, e.g. because it gives you a sentence that seems useful. You'll get stuck in Wonderland! *Don't make assumptions for $\forall$Elim or $\exists$Intro.*

2. When you make an assumption, always *write down your goal*. When the goal has been achieved, check it off and end the subproof. (Then try to remember why the heck you needed to prove that!)

3. Remember the batting order:

(i) Prepare for $\forall$Intro (as soon as your immediate goal is a $\forall$)
(ii) Prepare for $\exists$Elim (as soon as you have an $\exists$ to eliminate)
(iii) Use the constants introduced in (i) and (ii) to do the easy stuff, i.e. $\forall$Elim and $\exists$Intro.
(iv) If stuck, try $\neg$Intro (assume the negation of your goal)
(v) If still stuck, then PANIC!

4. Remember that each logical operator has an introduction and elimination rule. If you're trying to figure out how to use a sentence you know, quite likely you should use the elimination rule of its main operator. If you're trying to prove a certain goal, consider using the introduction rule of its main operator.

5. There are some logical rules, such as Quantifier de Morgan, that aren't included in the formal rules of our system $\mathcal{F}^{+}$. Feel free to use them, however, in figuring out *what is provable* from the information you have. Suppose you're trying to prove ⊥, for example. The problem with this is that you don't know *which* contradiction to aim for. Informal reasoning might help you identify a contradiction that's provable, so that you can aim for it.

6. *Work backwards*. If the conclusion follows, a formal proof *does* exist, and you *will* find it. Thus you can write down parts of the end of the proof, leaving a gap in the middle. This is almost always useful. The gap will be filled. In particular, work backwards from an existential. If a goal is ∃xP(x), for example, then ask yourself, "what thing can I prove to have the property P?" Quite often you'll know, or at least be able to guess, what object it is. Then (supposing the object is b) you'll write P(b) on the line above ∃xP(x).

7. Some types of sentence, such as ¬∃xP(x) and ¬∀xP(x) have no rule that allows them to be eliminated. I have referred to these as *uncrackable* premises. How then can you unlock the useful information they contain? In general you will use such a sentence with its negation to write down ⊥ by the rule ⊥Intro. (Deriving ⊥ is very useful in a reductio subproof, of course, as well as in proof by cases and ∃Elim.) You can even use an uncrackable sentence as a "crystal ball" to see some aspects of your future proof. If you already know the sentence ¬∀xP(x), for example, then you know that you will at some point prove ∀xP(x) to get a contradiction. Thus as soon as your goal is ⊥, and you have the resources to prove ∀xP(x), you might as well get on with it …