Philosophy 1102: Introduction to Logic

Department of Philosophy
Langara College

Really Useful Strategies for Boolean Proofs

1. Don't be in a hurry to start your proof. Think about it intuitively, asking yourself: "Is this really a TT consequence?" If it is, then sketch out an informal proof of it, in your head at least. If it seems invalid, then try to find a counterfactual world instead. (If such a world seems elusive, then go back to finding an informal proof!)

2. In constructing a proof, look first at each premise, and ask yourself "How can I eliminate this?" At this point you should *look at its main connective*. Remember that each logical operator has an introduction and elimination rule. If you're trying to figure out how to use a premise, or any known sentence, you should quite likely use the elimination rule of its main operator. You may find that some premises cannot be eliminated yet, for lack of additional information. In that case, make a note somewhere of the sentences you need, which you will likely prove (or perhaps assume) later.

3. *Before* you begin eliminating those premises, take a look at the conclusion. Ask yourself: "How can I introduce this?" At this point you should *look at its main connective* and consider using the introduction rule for that connective. This may require making an assumption, as described in #4.

4. If your immediate goal (i.e. within the present proof or subproof) is a negation, you will often use **$\neg$Intro**. If the goal is a conditional, you will almost always use **$\rightarrow$Intro**. These both require starting a new subproof, and setting a new goal, as follows:

(i) If you're trying to prove $\neg$P, by reductio/**$\neg$Intro**, assume P, and derive $\perp$.)
(ii) If you're trying to prove a conditional P $\rightarrow$ Q, using **$\rightarrow$Intro**, assume P and derive Q.

5. If you know that some disjunction like P $\vee$ Q, or P $\vee$ Q $\vee$ R, is true (either because it's a premise, or because you've already derived it) then you will almost always need to use **$\vee$Elim** (proof by cases) to extract the information from it. In the two/three subproofs for **$\vee$Elim** the goals are all the same, *and are the same as you previous goal*. Each subproof begins by assuming one of the two/three disjuncts, i.e. you assume P, derive the goal, then discharge that and assume Q, then derive the goal again. (And if there's a third disjunct you'll need to assume that as well and derive the goal a third time.) Finally you can "pull out" the goal from these subproofs and write it down below them.

6. *Only* make an assumption for one of these three reasons, as described in 4 and 5, i.e. for **¬Intro**, **→Intro** and **∨Elim**. *Never* (unless all else fails) make an assumption for other reasons, e.g. because it gives you something you want. You'll get stuck in Wonderland!

7. When you make an assumption, i.e. start a new subproof, always *write down your new goal* somewhere. Write it beside the assumption, labelled "goal" or "RTP", or better still in the subproof itself, as the last line. When the goal has been achieved, end the subproof. Unless you're in the middle of a proof by cases, you have now earned the right to believe something *outside* that subproof, so write that thing down.

8. There are some logical rules, such as de Morgan, that aren't included in the formal rules of our system $\mathcal{F}$. Feel free to use them, however, in figuring out *what is provable* from the information you have. Suppose you're trying to prove ⊥, for example. The problem with this is that you don't know *which* contradiction to aim for. Informal reasoning might help you identify a contradiction that's provable, so that you can aim for it.

9. Some types of sentence, such as ¬(P ∨ Q) and ¬(P ∧ Q), have no rule that allows them to be eliminated. I refer to these as *uncrackable* premises. How then can you unlock the useful information they contain? In general you will use such a sentence with its negation to write down ⊥ by the rule **⊥Intro**. (Deriving ⊥ is very useful in a reductio subproof, of course, as well as in proof by cases.) You can even use an uncrackable sentence as a "crystal ball" to see some aspects of your future proof: If you already know the sentence ¬(P ∨ Q), for example, then you know that you will at some point prove P ∨ Q to get a contradiction. Thus as soon as your goal is ⊥, and you have the resources to prove P ∨ Q, you might as well get on with it …

10. Think backwards. It's a good idea to write down your next goal in the proof itself, leaving plenty of space above to insert the necessary steps before it. Ask yourself: "What would I need to know in order to derive this?" If an idea comes to mind, then pencil that sentence in just above the goal. (Make sure that it is a TT consequence of the information above, however, or you will have accepted a "mission impossible"!) Repeat as necessary, i.e. continue to work backwards for as long as you seem to be getting somewhere. If you get stuck, try working forwards again.

11. As the therapists say, *pay attention to your needs*. As you're looking through the premises of the argument, or other sentences that you've already proved, you might find a conditional, e.g. (P ∨ Q) → ¬R. In that case, you say to yourself: "I need to know (P ∨ Q) in order to eliminate this." Make a note of it somewhere, and take steps to meet this need.

12. Most importantly: HAVE FUN!