Translations Involving Complex Quantifiers

1. Translation of definite descriptions using Russell's method.

A definite description, in English, is an expression of the form

    the F

where F is a one-place predicate. Some examples of definite descriptions are

    the present Prime Minister of Canada

    the last team the Canucks beat

    the cube that is in the same row as b

    the tallest student in the class

A definite description "the F" is not a sentence, of course. To form a sentence from a definite description, one must ascribe some property to the unique thing that is F. Thus the general form of a sentence involving a definite description is:

    the F is G

 where G is another one-place predicate. Some examples of sentences of this form are:

    I like the present Prime Minister of Canada.

    The last team the Canucks beat was disorganised.

    The cube that is in the same row as b is large.

    Karen sits next to the tallest student in the class.

According to Russell, the sentence "the F is G" really says that there exists one and only one thing with the property F, and that thing also has the property G. In FOL this is written:

$$\exists x(F(x) \wedge \forall y(F(y) \rightarrow x = y) \wedge G(x)).$$

So, the sentence The cube that is in the same row as b is large is written:

(A) $\exists x((Cube(x) \wedge SameRow(x, b)) \wedge \forall y((Cube(y) \wedge SameRow(y, b)) \rightarrow x = y) \wedge Large(x)).$

Note that 'F( )' has been replaced throughout by 'Cube( ) $\wedge$ SameRow(, b)', and 'G( )' has been replaced by 'Large( )'.

Sometimes it's tricky to tell if a property mentioned in the English sentence is part of the F or the G property. Consider, for example,

The cube, which is in the same row as b, is large.

This may look equivalent to the sentence above, but it isn't. This one says that there exists one and only one cube in the world, *and* it is in the same row as b, *and* it is large. In FOL we can write this as:

(B) $\exists x(Cube(x) \wedge \forall y(Cube(y) \rightarrow x = y) \wedge SameRow(x, b) \wedge Large(x)).$

Note that this is a stronger proposition than "the cube that is in the same row as b is large", as (B) entails (A). As an exercise, you can show that (A) does not entail (B) by constructing a world where (A) is true but (B) is false.

Now consider the sentence:

Karen sits next to the tallest student in the class.

We could translate this one in exactly the same way, by introducing a predicate such as Tallest(x), which says that x is a tallest student in the class. We can do much better than this, however, by *constructing* this predicate from the predicates Taller(x, y) (x is taller than y) and InClass(x) (x is a student in the class). What does it mean to say that x is a tallest student in the class? In means that x is a student in the class, and x is taller than every other student in the class. In FOL this is written:

$InClass(x) \wedge \forall y((InClass(y) \wedge x \neq y) \rightarrow Taller(x, y))$

Using this to translate 'x is a tallest student in the class', the above sentence becomes:

$\exists x(InClass(x) \wedge \forall y((InClass(y) \wedge x \neq y) \rightarrow Taller(x, y)) \wedge SitsNextTo(karen, x))$

It may seem that we've missed out the middle part of this translation, the part that says that *only* x has the property F. But there's no need to say this, for the very meaning of this F guarantees that one thing (at most) can have it. There cannot exist two students in the class, each of whom is taller than everyone else in the class, for then each of these two would be taller than the other.

Frege's method

Sometimes the phrase "the F" involves a function. E.g. "the father of Plato was bald". Then it's just:

Bald(father(plato))

2. Numerical (Existential) Quantifiers

The standard existential quantifier says that there is *at least one* object in the domain such that …. What if we want to say that there are at least *two* objects, however? Or exactly five objects? Or three at most? We could introduce a new quantifier for each of these, but it is actually possible to express these claims using the standard two quantifiers. The only drawback is that, for numbers larger than two, the sentences become rather long and complex!

2.1 *At Least* Quantifiers

The simplest kind of numerical quantifier says that there are *at least n* objects such that … For example, suppose we want to say that there are at least four large things in the world. Basically we say that there is a thing, there's a thing, there's a thing, and there's a thing, such that no two of them are identical, and all of them are large. In FOL this is written:

$$\exists x \exists y \exists z \exists w (x \neq y \wedge x \neq z \wedge x \neq w \wedge y \neq z \wedge y \neq w \wedge z \neq w \wedge Large(x) \wedge Large(y) \wedge Large(z) \wedge Large(w))$$

You see that, even with just four things, it's quite a lot of work to say that no two of them are identical!

2.2 *Exactly* Quantifiers

To say that there are exactly *n* (objects), i.e. no more and no less than *n*, is more tricky. We first have to say that there are at least *n* objects, as in Section 2.1, and then we have to add that there are no more than *n*. In other words, we add that any further object with the stated property is identical to one of the objects already mentioned. So here's how to say that there are exactly two large things in FOL.

$$\exists x \exists y (x \neq y \wedge Large(x) \wedge Large(y) \wedge \forall z(Large(z) \rightarrow (z = x \vee z = y)))$$

2.3 *At Most* **Quantifiers**

The final kind of numerical quantifier says that there are *at most n* object such that … In other words, the number of such objects is less than or equal to *n*. (Note that this allows there to be no object at all with that property.) This quantifier is a little trickier than the others, so let's start with the simplest case, i.e. "there is at most *one* large thing". Perhaps the most obvious way to express this is by saying "There do not exist two (non-identical) large things", which in FOL is:

$$\neg \exists x \exists y (x \neq y \wedge \text{Large}(x) \wedge \text{Large}(y))$$

You see that the trick is basically to deny the existence of more than one large thing, i.e. to deny the existence of at least two large things. Note that this automatically rules out the existence of three large things, four large things, and so on, for in those cases there will exist at least two large things. Another way to make the same claim is to say "Any two large things are identical", i.e.

$$\forall x \forall y ((\text{Large}(x) \wedge \text{Large}(y)) \rightarrow x = y)$$

In general, to say that there are at most *n* things is to claim that it's not the case that there are at least (*n* + 1) things. The easiest way to say this is to use the second form, with the universal quantifiers, i.e. "Take any (*n* + 1) objects. Then at least one pair of these are identical." In the case of "There are at most four large things", this becomes:

$$\forall v \forall w \forall x \forall y \forall z ((\text{Large}(v) \wedge \text{Large}(w) \wedge \text{Large}(x) \wedge \text{Large}(y) \wedge \text{Large}(z)) \rightarrow (v=w$$
$$\vee \; v=x \vee v=y \vee v=z \vee w=x \vee w=y \vee w=z \vee x=y \vee x=z \vee y=z))$$

Not very pretty!